ON MATCHING EXTENSIONS WITH PRESCRIBED AND PROSCRIBED EDGE SETS II

by

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ABSTRACT

Let G be a graph with at least $2(m+n+1)$ vertices. Then G is $E(m,n)$ if for each pair of disjoint matchings $M, N \subseteq E(G)$ of size m and n respectively, there exists a perfect matching F in G such that $M \subseteq F$ and $F \cap N = \emptyset$. In this paper, we extend previous results due to Chen [C] as well as results of the present authors concerning the property $E(m,n)$. The first extends the class of claw-free graphs and the second generalizes a result about bipartite graphs.

1. Introduction

In this paper all graphs will be finite and, unless otherwise specified, simple as well. Let G be a graph with at least $2(m+n+1)$ vertices. G is said to be $E(m,n)$ if for every pair of disjoint matchings $M, N \subseteq E(G)$ there is a perfect matching F in G such that $M \subseteq F$ and $F \cap N = \emptyset$. If G is $E(n,0)$ we say that G is n-**extendable**. In fact, it was the concept of n-extendability which subsequently gave rise to the property $E(m,n)$. Graphs which are n-extendable have been studied quite extensively. (See [P4] and [P6].) Some of the early results on this family of graphs are also to be found in the book [LP] where their connection with other areas of matching theory are also discussed. For further information on n-extendable graphs, we refer the interested reader to these three sources and the reference lists contained therein.

The first paper to treat the more general concept of $E(m,n)$ was due to Porteous and one of the present authors [PA]. In this paper the general theme is the study of when the implication $E(m,n) \longrightarrow E(p,q)$ holds and does not hold. Although it has long been known that n-extendability implies $(n-1)$-extendability [P1], there are a few surprises in the implication lattice for the general case $E(m,n)$.

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2. A result for $K_{1,r}$-free graphs

C. Chen [C], and independently one of the present authors [P7], have proved the following theorem.

Theorem 2.1. Let $m \geq 1$, $r \geq 2$ and let G be a $(2m+r-2)$-connected $K_{1,r}$-free graph of even order at least $2m+2$. Then G is $E(m,0)$.

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We extend this result in the theorem below.

Theorem 2.2. Let m, n and r be non-negative integers with $m \geq 1$, $r \geq 3$ and let G be a $(2m+n+r-2)$-connected $K_{1,r}$-free graph of even order at least $2m+2n+2$. Then G is $E(m,n)$.

Proof. The proof is by induction on n. When $n = 0$, the result is true by Theorem 2.1. Suppose k is the smallest integer such that there exists a graph G which is $(2m+k+r-2)$-connected even, $K_{1,r}$-free but not $E(m,k)$. Hence there exist matchings $M = \{e_1, \ldots, e_m\}$ and $K = \{f_1, \ldots, f_k\}$ such that graph $G' = G - V(M) - K$ has no perfect matching. Thus by Tutte's theorem on perfect matchings, and since G is even, there exists a set $S' \subseteq V(G')$ such that $c_o(G'-S') \geq |S'|+2$.

By the inductive hypothesis, G is $E(m, k-1)$ and thus each f_i joins two different odd components of $G'-S'$ and so $c_o(G'-S') = |S'|+2$. Denote these odd components by $C_1, \ldots, C_{|S'|+2}$.

In the graph G shrink the subgraphs corresponding to $C_i, i = 1, \ldots, |S'|+2$, each to a single vertex, v_i, to form a new graph G'' where we suppress any parallel edges or loops thus formed. Let N denote the number of edges in G'' joining the vertices of $S' \cup V(M)$ to the $|S'|+2$ different v_i's.

Claim. $|S'| \geq k$ (and thus $S' \cup V(M)$ is a cutset in G).

Suppose, by way of contradiction, that $|S'| \leq k-1$. Let D denote the subgraph of G'' induced by $\{v_i : 1 \leq i \leq |S'|+2\}$. Now adding the k edges $f_1, \ldots, f_k$ to the graph D yields a (multi)graph G^* which is at most $(k-|S'|)$-edge-connected. A cutset in G can be formed from $S' \cup V(M)$ together with one endvertex (appropriately chosen in G) of each edge in a minimum edge cutset in G^*. (Note that this procedure could fail in the case when $|S'| = 0$ and $k = 1$. However, such failure would indicate that the two odd components of $G - V(M)$ were singletons and that $|V(G)| = 2m+2$, contradicting the minimum order condition in the statement of the theorem.) Such a cutset L has at most $2m + |S'| + k - |S'| = 2m+k$ vertices. Since G is $(2m+k+r-2)$-connected, this yields $2m+k \geq 2m+k+r-2$ or $r \leq 2$, a contradiction and the claim is proved.

As a result of the claim, we see that $G - V(M) - S'$ is disconnected and thus $|V(M)| + |S'| \geq 2m+k+r-2$: that is, $|S'| \geq k+r-2$.

Denote by N the number of edges joining $V(D)$ to $V(M) \cup S'$. Since G is $(2m+k+r-2)$-connected, each vertex in $V(D)$ has degree at least $2m+k+r-2$. There are k edges between the v_i's, so

$$N \geq (|S'|+2)(2m+k+r-2) - 2k.$$

We now wish to bound N above. Let us view N from $S' \cup V(M)$. We distinguish two cases.

First suppose $k < r$. Let $u \in V(M) \cup S'$ and suppose it is adjacent to both ends of $f_i, i = 1, \ldots, j$. Then, since G is $K_{1,r}$-free, vertex u has at most $r - j - 1$ other neighbors in $\{v_1, \ldots, v_{|S'|+2}\}$. But then vertex u has at most $2j + (r - j - 1) = j + r - 1 \leq k + r - 1$ neighbors in $\{v_1, \ldots, v_{|S'|+2}\}$. Thus $N \leq (2m + |S'|)(k + r - 1)$.

So $(2m + |S'|)(k + r - 1) \geq N \geq (|S'| + 2)(2m + k + r - 2) - 2k = (|S'| + 2)(2m + (k + r - 1) - 1) - 2k$. Subtracting $|S'|(k + r - 1)$ from both sides, we get

$$2m(k + r - 1) \geq 2m|S'| - |S'| + 4m + 2r - |S'| - 4. \tag{2.1}$$

Now, $|S'| \geq k + r - 2$. Substituting this bound for $|S'|$ into the corresponding positive term in the right-hand side of (2.1) one obtains

$$\begin{aligned} 2m(k + r - 1) &\geq 2m((k + r - 1) - 1) + 4m + 2r - |S'| - 4 \\ \text{i.e.} \qquad |S'| &\geq 2m + 2r - 4 \geq 2m + k + r - 3, \end{aligned} \tag{2.2}$$

since $k < r$. Substituting this new bound for $|S'|$ from (2.2) into the corresponding positive term in the right-hand side of (2.1) one obtains

$$\begin{aligned} 2m(k + r - 1) &\geq 2m(2m + (k + r - 1) - 2) + 4m + 2r - |S'| - 4 \\ \text{i.e.} \qquad |S'| &\geq (2m)^2 + 2r - 4 \geq (2m)^2 + k + r - 3, \end{aligned} \tag{2.3}$$

again using the assumption that $k < r$. Substituting this new bound for $|S'|$ from (2.3) into the corresponding positive term in the right-hand side of (2.1) one gets

$$\begin{aligned} 2m(k + r - 1) &\geq 2m((2m)^2 + (k + r - 1) - 2) + 4m + 2r - |S'| - 4 \\ \text{i.e.} \qquad |S'| &\geq (2m)^3 + 2r - 4 \geq (2m)^3 + k + r - 3. \end{aligned} \tag{2.4}$$

Continuing in this way we find, after substituting the bound for $|S'|$ from (2.j) into the corresponding positive term in the right-hand side of (2.1), that

$$\begin{aligned} 2m(k + r - 1) &\geq 2m((2m)^j + (k + r - 1) - 2) + 4m + 2r - |S'| - 4 \\ \text{i.e.} \qquad |S'| &\geq (2m)^j + 2r - 4 \geq (2m)^j + k + r - 3. \end{aligned} \tag{2.$j+1$}$$

Thus, since $m \geq 1$, $|S'|$ is unbounded above, contradicting the finiteness of G. Hence we may assume that $k \geq r$. Remembering that G is $K_{1,r}$-free, we have

$$\begin{aligned} (|S'| + 2m)(2(r - 1)) &\geq (|S'| + 2)(2m + k + r - 2) - 2k \\ &= (|S'| + 2)(2m + 2(r - 1) + (k - r)) - 2k. \end{aligned}$$

Subtracting $|S'|(2(r - 1))$ from both sides of the inequality we get

$$2m(2(r - 1)) \geq |S'|(2m + (k - r)) + 4m + 2r - 4.$$

Adding $4m$ to both sides, the inequality becomes

$$\begin{aligned}
4mr &\geq |S'|(2m + (k - r)) + 8m + 2r - 4 \\
&= 2m|S'| + |S'|(k - r) + 8m + 2r - 4 \\
&\geq 2m(k + r - 2) + |S'|(k - r) + 8m + 2r - 4 \\
&\geq 2m(2r - 2) + |S'|(k - r) + 8m + 2r - 4 \text{ (since } k \geq r) \\
\text{i.e.} \qquad 0 &\geq |S'|(k - r) + 4m + 2r - 4 \geq 4m + 2 \text{ (since } r \geq 3).
\end{aligned}$$

The right-hand side of this last inequality is strictly positive and consequently we have reached a contradiction. The proof of the theorem is now complete.

∎

If $n \leq r-1$, then the connectivity hypothesis in the preceding theorem is sharp in the sense that it is easy to construct an even graph G which is $(2m+n+r-3)$-connected and $K_{1,r}$-free but which is not $E(m,n)$. To construct such a G, let H_1 be the complete graph $K_{2m+n=r-3}$ and let H_2 consist of n independent edges and an additional $r-1-n$ is isolated vertices. Then let $G = H_1 + H_2$. It is easy to verify that G is $(2m+n+r-3)$-connected and $K_{1,r}$-free. Let M be any set of m independent edges in H_1 and N be the n independent edges in H_2. The reader can easily verify that there exists no perfect matching in G containing M and avoiding N. So G is not $E(m,n)$.

If $n \geq r$ we do not know if the conclusion of Theorem 2.2 holds if the connectivity hypothesis is reduced by 1.

Sumner [S] proved the following result.

Theorem 2.3. If $r \geq 3$ and G is $(r-1)$-connected, $K_{1,r}$-free and even, then G contains a perfect matching.

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Furthermore, for $r = 3$, Sumner [S], and independently Las Vergnas [LasV], also obtained the following stronger result.

Theorem 2.4. If G is connected, claw-free and even, then G contains a perfect matching.

∎

In light of Theorem 2.3 it is tempting to conjecture that Theorem 2.4 can be improved to state that every $(r-2)$-connected, $K_{1,r}$-free even graph contains a perfect matching, when $r \geq 4$. But this is false for all $r \geq 4$. We now present counterexamples for all such r.

For $r = 4$, let $,_4$ be the 10 vertex graph obtained from K_4 by subdividing each edge with a single vertex. For $,_r$, $r \geq 5$, we first state and prove the following lemma.

Lemma 2.5. For all $r \geq 5$, there exists a graph G_r which is $(r-2)$-connected, $(r-2)$-regular and bipartite and which has $2(2r-4) = 4r-8$ vertices.

Proof. Our construction is inductive. For $r = 5$, let G_5 be $C_6 \times K_2$ (i.e. the hexagonal prism).

Assume that for all r, $5 \leq r < k$, we have constructed G_r. Since G_{k-1} is bipartite and regular, by König's Edge-Coloring Theorem it must contain a perfect matching M_{k-1}. Let the vertices of G_{k-1} be labelled such that the matching $M_{k-1} = \{b_i w_i | 1 \leq i \leq 2k-6\}$. To construct G_k from G_{k-1} we use four new vertices $U = \{b, b', w, w'\}$, together with the new edges obtained by joining b to $w_1, \ldots, w_{k-3}$, b' to $w_{k-2}, \ldots, w_{2k-6}$, w to $b_{k-2}, \ldots, b_{2k-6}$, w' to $b_1, \ldots, b_{k-3}$, b to w and b' to w'. The graph G_k is illustrated in Figure 1.

Figure 1.

It is obvious that G_k is bipartite, $(k-2)$-regular and has $2(2k-4)$ vertices.

It remains to show that G_k is $(k-2)$-connected. Let S be a minimum cutset in G_k and assume, to the contrary, that $|S| \leq k-3$. Let $S' = S \cap V(G_{k-1})$.

First assume that $G_{k-1} - S'$ is connected. If each $u \in U$ has a neighbor in $G_{k-1} - S'$, then $G_k - S$ is connected, a contradiction. So without loss of generality we may assume that $S' = N_{G_k}(b) = \{w_1, \ldots, w_{k-3}\}$. Hence $S = S'$. But then, $G_k - S - b$ is connected and therefore, since b is adjacent to w, $G_k - S$ is connected as well. But this is a contradiction.

So we may assume that $G_{k-1} - S'$ is not connected. But then since G_{k-1} is $(k-3)$-connected and $|S| \leq k-3$, we must have $k-3 \leq |S'| \leq |S| \leq k-3$ and hence $S = S'$ and $|S| = k-3$. So $S \subseteq V(G_{k-1})$. But if H_k is the spanning subgraph of G_k having as its edge set M_{k-1} together with all the edges incident with vertices in U, it is clear that H_k – and hence G_k – cannot be separated by any subset of $V(G_{k-1})$ of size $k-3$. This contradiction establishes the lemma.

∎

We now proceed to construct , $_r$, $r \geq 5$. Let $B \cup W$ be the bipartition of G_r, where B denotes the set of $2r-4$ "black" vertices and W, the set of $2r-4$ "white" vertices. (See again Figure 1.) Let β_1 and β_2 be two new "black" vertices. Join β_1 to half the white vertices in G_r and β_2 to the other half. Let the resulting graph be , $_r$. Then deg $\beta_i = r-2$, for $i = 1, 2$ and it follows that , $_r$ is $(r-2)$-connected. Moreover , $_r$ has maximum degree $r-1$ and hence , $_r$ must be $K_{1,r}$-free. On the other hand, , $_r$ is an unbalanced bipartite graph and hence has no perfect matching.

3. A result for bipartite graphs

REWRITE FOLLOWING PARAGRAPH!!!!!!!!!!!!!!!!!!!!!!!!

In this section we shall generalize a result on extending matchings in regular bipartite graphs first obtained in [AHP^2].

Before presenting this result, we state a lemma the proof of which is self-evident.

Lemma 3.1. Let n and r be non-negative integers. Let G be an r-regular bipartite graph with $r \geq n+1$. Then G is $E(0, n)$.

Proof. Let $f_1, \ldots, f_n$ be a set of n independent edges in G. Since G is a regular bipartite graph, by König's Theorem, $E(G)$ may be partitioned into r perfect matchings. Now, $r \geq n+1$ so there exists a perfect matching of G which avoids all of the edges $f_1, \ldots, f_n$.

∎

Lemma 3.2. If G is an r-regular graph with cyclic connectivity, $c_\lambda(G) \geq r \geq 2$, then G is r-edge-connected.

Proof. Suppose G has an edge cut L, with $|L| < r$. Then at least one component of $G - L$ is a tree. Let this component be T.

If $|V(T) = 1$, then $|L| = 1$, a contradiction. So $|V(T) \geq 2$ and hence tree T has at least two endvertices. Suppose some edge of L joins two vertices of T. Then $G - L - V(T)$ is acyclic, since L is not *cyclic* edge cut. So let T' be one of the components of $G - l - V(T)$. Then T' is a tree and furthermore no two of its vertices are joined by an edge of L. Again, T' is not a single vertex, so T' has at least two endvertices. Each of these two endvertices of T' is incident with exactly $r-1$ edges of L. Hence $|L| \geq 2(r-1)$. But $2(r-1) \geq r$ since $r \geq 2$. So $|L| \geq r$, a contradiction.

∎

Lemma 3.3. Let F be a forest with no isolated vertices. Suppose the bipartition of $V(F)$ is $A \cup B$, where $|A| = a, |B| = b$ and $a > b$. Then F contains a tree with at least two endvertices in A.

∎

We are now prepared to state and prove the main result of this section.

Theorem 3.4. Let m, n and r be non-negative integers with $r > \max\{2n, (m/2)+2\}$. Let G be an r-regular bipartite graph with $|V(G)| \geq 2m+2n+2$ and

$$c_\lambda(G) \geq \begin{cases} 0, & \text{when } m = 0, \\ 2n+1, & \text{when } m = 1, \text{ and} \\ (m-1)r+n+1, & \text{for all } m \geq 2. \end{cases}$$

Then G is $E(m,n)$.

Proof. Note that $r \neq 1$. Moreover, if $r = 2$, then $m = 0$ and $n = 0$. But in this case, by König's Edge-Coloring Theorem, G contains a perfect matching, i.e. G is $E(0,0)$. In consequence, we shall assume henceforth that $r \geq 3$.

When $m = 0$, since $r \geq 2n+1 \geq n+1$, the result follows immediately as in the prrof of Lemma 3.1.(In fact, Lemma 3.1 is stronger in that it requires a less stringent lower bound on r.)

When $m = 1$, since $c_\lambda(G) \geq 2n+1$ and $r \geq 2n+1$, we claim that G is (2n+1)-edge-connected. To see this let L be a minimum edge cut in G. If L is a *cyclic* edge cut, we are done. So suppose $G - L$ has a component T which is a tree. If T is a single vertex, $|L| = r \geq 2n+1$, by the definition of r and again we are done. Hence suppose that T contains at least two vertices and therefore at least two endvertices. It follows that $|L| \geq 2(r-1) \geq r \geq 2n+1$, since $r \geq 3$. Thus by Theorem 3.1 of [AHP^2], G is $E(1,n)$.

For the remainder of the proof, we shall assume that $m \geq 2$. We shall proceed by induction on n, noting that when $n = 0$, the result follows immediately from Theorem 2.2 of [P3]. Assume that for all values of $n < k$, the theorem holds and suppose that the theorem fails for $n = k$; i.e., let G be an r-regular bipartite graph with $|V(G)| \geq 2m+2k+2$, $r \geq \max\{2k+1, m+2\}$ and $c_\lambda(G) \geq (m-1)r+k+1$ and suppose that G is not $E(m,k)$. So there exist two sets of independent edges, $M = \{e_1, \ldots, e_m\}$, $K = \{f_1, \ldots, f_k\}$ and $M \cap K = \emptyset$, such that $G' = G - V(M) - K$ contains no perfect matching. Let $A \cup B$ be the bipartition of $V(G)$ and for $i = 1, \ldots, m$, let $e_i = a_i b_i$, where $a_i \in A$ and $b_i \in B$. Then by the bipartite matching theorem of Philip Hall, we may assume, without loss of generality, that there exists a vertex set $A_1 \subseteq A$ with neighborhood $B_1 \subseteq B$ in G' such that $|A_1| > |B_1|$. Let A_2 be the set consisting of $\{a_1, a_2, \ldots, a_m\}$ and let B_2 be the set consisting of $\{b_1, b_2, \ldots, b_m\}$. finally, let $A_0 = A - (A_1 \cup A_2)$ and let $B_2 = B - (B_1 \cup B_2)$.

By the choice of k, G is $E(m, k-1)$ and hence $G' \cup f_i$ contains a perfect matching for each $i = 1, \ldots, k$. Hence, each f_i must join a vertex in A_1 to a vertex of B_0. Furthermore, $|A_1| = |B_1| + 1$ and $|B_0| = |A_0| + 1$. We denote by G_i the subgraph of G' induced by $A_i \cup B_i, i = 0, 1, 2$. We note that, since the degree of every vertex in A_1 is at least $m+2$, since K is a matching, and since $|B_2| = m$, it follows that the set B_1 is not empty. Furthermore, by a symmetric argument, since $B_0 \neq \emptyset$, $A_0 \neq \emptyset$.

For the rest of this proof, we adopt the following terminology:

$$\begin{aligned} q = &\ \text{the number of edges from } A_1 \text{ to } B_2, \\ n_0 = &\ \text{the number of edges from } B_0 \text{ to } A_2, \\ n_1 = &\ \text{the number of edges from } A_0 \text{ to } B_1, \\ n_2 = &\ \text{the number of edges from } A_2 \text{ to } B_1, \\ n_3 = &\ \text{the number of edges from } B_2 \text{ to } A_0, \text{ and} \\ n_4 = &\ \text{the number of edges in } G_2. \end{aligned}$$

(See Figure 2.)

Figure 2.

Counting edges in G_1, we have $|A_1|r - q - k = |B_1|r - n_1 - n_2$, or

$$|A_1|r = |B_1|r - n_1 - n_2 + q + k. \tag{3.1}$$

Also $|A_1| = |B_1| + 1$, so

$$|A_1|r = (|B_1| + 1)r. \tag{3.2}$$

Combining (1) and (2) we get $(|B_1| + 1)r = |A_1|r = |B_1|r - n_1 - n_2 + q + k$, or

$$k + q = n_1 + n_2 + r. \tag{3.3}$$

Claim 1. If G_0 is acyclic, then $|A_0| + |B_0| \leq 2m + 1$.

Since G_0 is a forest, $|V(G_0)| \geq |E(G_0)| + 1$, and so

$$\begin{aligned}
2(|V(G_0)|) &= 2(|A_0| + |B_0|) \\
&\geq 2(|E(G_0)| + 1 \\
&= (\sum_{v \in V(G_0)} \deg_{G_0} v) + 2 \\
&= |A_0|r - n_1 - n_3 + |B_0|r - n_0 - k + 2
\end{aligned}$$

or

$$(r-2)(|A_0| + |B_0|) \leq n_0 + n_1 + n_3 + k - 2. \tag{3.4}$$

Now counting edges out of A_2 in G, we also have

$$rm = n_0 + n_2 + n_4 \geq n_0 + n_2 + m$$

so

$$n_0 + n_2 \leq (r-1)m$$

and hence

$$n_0 \leq (r-1)m - n_2, \tag{3.5}$$

and similarly, counting edges out of B_2 in G, we also have

$$n_3 \leq (r-1)m - q. \tag{3.6}$$

Substituting (3.5), (3.3) and 3.6) into (3.4), we get

$$\begin{aligned}
(r-2)(|A_0| + |B_0|) &\leq k + (r-1)m - n_2 + n_1 + n_3 - 2 \\
&= k + (r-1)m + n_1 + n_2 - 2n_2 + n_3 - 2 \\
&\leq k + (r-1)m + k + q - r - 2n_2 + n_3 - 2 \\
&\leq 2k + (r-1)m + q - r - 2n_2 + (r-1)m - q - 2 \\
&= 2k + 2(r-1)m - r - 2n_2 - 2 \\
&\leq 2k + 2(r-1)m - r - 2.
\end{aligned}$$

Noting that $r - 2 > 0$ and dividing both sides by $r - 2$, we obtain

$$\begin{aligned}
(|A_0| + |B_0|) &\leq \frac{2k + 2(r-1)m}{r-2} - \frac{r+2}{r-2} \\
&< \frac{2m(r-1)}{r-2} + \frac{2k}{r-2} - \frac{r-1}{r-2} \\
&= (2m-1)(\frac{r-1}{r-2}) + \frac{2k}{r-2} \\
&= (2m-1)(1 + \frac{1}{r-2}) + \frac{2k}{r-2} \\
&= (2m-1) + \frac{2m+1}{r-2} + \frac{2k}{r-2} \\
&\leq (2m-1) + 2 - \frac{1}{m} + 1 + \frac{1}{2k-1} \\
&= 2m + 2 + (\frac{1}{2k-1} - \frac{1}{m}).
\end{aligned}$$

Now $1/(2k-1) < 1$ for all $k \geq 0$, so, $1/(2k-1) - 1/m < 1$. So $|A_0| + |B_0| < 2m + 2 + 1 = 2m + 3$. But $|A_0| + |A_0|$ is odd, so $|A_0| + |B_0| \leq 2m + 1$, and thus Claim 1 is proved.

Claim 2. $n_1 + n_2 + n_3 + n_4 \leq r(m-1) + k$.

To see this, note that

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 &= n_1 + n_2 + n_3 + (rm - q - n_3) \\ &\leq k + q - r + n_3 + (rm - q - n_3) \\ &= r(m-1) + k. \end{aligned}$$

We now define $H_0 = G[A_0 \cup B_0 \cup A_2]$ and $H_1 = G[A_1 \cup B_1 \cup B_2]$.

Claim 3. Subgraph H_1 contains a cycle.

Suppose not. Then H_1 is a forest and hence so is G_1. Since $B_1 \neq \emptyset$, G_1 contains no isolates. But $|A_1| > |B_1|$ and so, by Lemma 3.3, G_1 contains a tree T_1 with at least two endvertices in A_1. On the other hand, since G is r-regular and $r > (m/2) + 2$, every two such endvertices must share a common neighbor in B_2 and hence, since $m \geq 2$, H_1 contains a cycle; a contradiction.

Claim 4. H_0 contains a cycle.

Suppose not. Then H_0 is a forest and hence if $|B_0| = \ell$ (and thus $|A_0| = \ell - 1$), then

$$\begin{aligned} |V(H_0)| &= |A_0| + |B_0| + m \\ &= 2\ell - 1 + m \\ &\geq |E(H_0))| + 1 \\ &= (m + \ell - 1)r - (n_1 + n_2 + n_3 + n_4) + 1 \\ &\geq (m + \ell - 1)r - (r(m-1) + k) + 1 \\ &= \ell r - k + 1. \end{aligned}$$

So, in particular, $\ell r - k + 1 \leq 2\ell - 1 + m$ and hence

$$\ell r \leq \ell + m + k - 2. \tag{3.7}$$

Hence $\ell(m+2) \leq \ell r \leq 2\ell + m + k - 2$ and therefore

$$\ell m \leq m + k - 2. \tag{3.8}$$

By Claim 1, $2\ell - 1 \leq 2m + 1$ and hence

$$\ell \leq m + 1. \tag{3.9}$$

On the other hand, since the edges in K are independent, $\ell \geq k$, and so, (3.8)

$$\ell m \leq m + \ell - 2. \tag{3.10}$$

Substituting (3.9) into (3.10) we obtain $\ell m \leq m+(m+1)-2 = 2m-1$. But $\ell \geq 2$. Hence $2m \leq \ell m \leq 2m-1$, a contradiction which completes the proof.

∎.

We note that Theorem 3.4 does not apply to 2-regular bipartite graphs. But clearly all such graphs have perfect matchings and hence are $E(0,0)$. We also note that, in the special case when $m = 1$ and $r = 2n+1$, there exist r-regular bipartite graphs with at least $2n+4$ vertices and cyclic connectivity at least $2n+1$, but which are not $E(2,n)$ as well as other such graphs which are not $E(1,n+1)$. For examples of both types, the reader is referred to [AHP²].

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