

# Hydroelastic Perspectives of Ocean Wave / Sea Ice Connectivity I

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**Abstract.** Global climate change has had a major influence on the sea ice canopy of the polar regions, which is expected to continue into the foreseeable future. The upward-trending incidence of storms and associated increases in winds and significant wave heights are now known to be contributing to this change, by breaking up the sea ice into smaller, more mobile pieces that can easily disperse under the action of off-ice winds. In this first of two interrelated papers we investigate breakup from the perspective of defining the conditions under which sea ice will fracture. We recommend using a flexural strength value that takes account of anelasticity, the strain rates involved and, crucially, how material moduli vary from the top to the underside of each ice plate. An Euler-Bernoulli beam and a Kirchhoff-Love plate are considered, as the assumption of thinness is defensible because waves and swell are comparatively long.

**Key words:** *Polar sea ice; ocean waves and swell; fracture and breakup.*

## 1. Introduction

### 1.1. Motivation

Compared with earlier decades, Arctic sea ice has thinned, adapted from predominantly perennial ice to seasonal ice and reduced in extent to nearly 30% less at the end of a longer summer melt period. Indeed, in recent years, the seasonal ice retreat has expanded dramatically [1, 2, 3], leaving much of the Beaufort Sea ice free at the end of summer [4]. Conversely, Antarctic sea ice is stable or increasing slightly [5]. Concomitantly, the influence of ocean waves is greater than ever before, both due to a growth of storm activity associated with climate change, leading to heightened winds and significant wave heights at high latitudes especially [6], and the creation of more fetch as sea ice fields become less concentrated with thinner ice floes present on average [4]. While somewhat neglected in ice/ocean models and climate models in the past, including when forecasting, the importance of wave-ice interactions has recently been recognized

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by agencies with polar interests such as the US Office of Naval Research, which has embarked on a significant departmental research initiative to collect and interpret *in situ* field data and to create models at various scales.

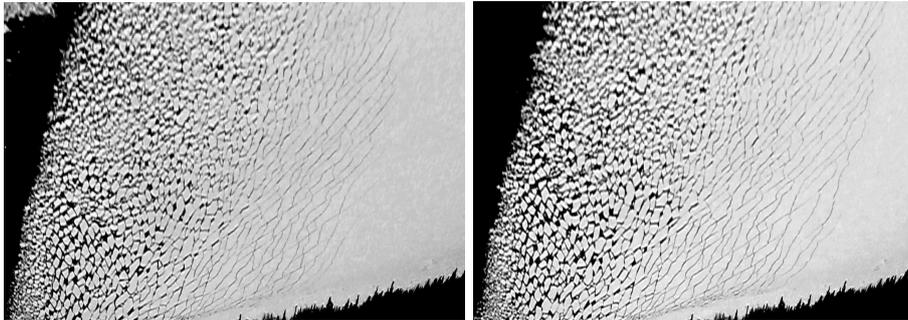
This paper is the first of two that consider how sea ice breaks up due to the action of ocean waves, a topic that will be new to many of the participants at the 7<sup>th</sup> International Conference on Hydroelasticity in Marine Technology, yet one that has a remarkable affinity to the conference theme in regard to both mathematical modelling and the gathering and analysis of experimental data [7]. It is a fact, for example, that the same or at least equivalent mathematical methods are being deployed in both fields and that, especially for pontoon type VLFS (very large floating structures), the physical and geometrical parameters are not dissimilar. On the other hand, we also recognize that some familiarization of sea ice physiognomy will be necessary in order for the reader to understand the rationale behind the study. This is provided here.

## 1.2. Sea ice 101: An abecedarian's introduction to sea ice

Sea ice is frozen sea water. In calm, sheltered conditions it typically grows to a metre or two in thickness during the winter months to form a continuous sheet of ice at the surface. It is a two-phase, two-component, reactive porous medium; an example of what is known in other contexts as a mushy layer [8]. Invariably, winds and currents move the plates of sea ice around, sometimes causing vast ice floes to crack and separate to create a so-called lead, floes to raft over one another, or pressure ridges (hummocks and keels) to be created in shear and/or compression from the typically thinner ice that forms in leads once they are opened up to cold air. So, away from the action of ocean waves, sea ice can be thought of as a vast sheet composed of many smaller plates either welded together or separated by cracks, with a complex topography on the surface and underside created by sinuous ridges that weave their way across the ice terrain. This type of ice is customarily called pack ice.

Although ice is one of the most common materials on Earth, it is very different from all other known materials; it may behave as an elastic, brittle, viscoelastic or even as a quasi-liquid material depending on its morphology and microstructure. It is not expected that a satisfactory parsimonious model of ice will be found across all environmental time and space scales. Sea ice is an heterogeneous mushy aggregate in which fluid (liquid and gas) inclusions are present that coexist with a large temperature gradient in nature because it is floating in water of about  $-1.8^\circ\text{C}$  but normally has much colder air against its upper surface. Nonuniform stress fields are created under flexural loads that require assumptions about its material behaviour, suggesting that in some situations its macroscopic behaviour cannot be understood without knowing its microstructure.

When ocean waves are present, they can travel into the ice pack beneath the ice floes from which it is composed, so that each floe effectively behaves as a floating raft which also flexes significantly on the wave's profile [9, 10]. The region where waves influence the ice cover is usually called the marginal ice zone (MIZ). The MIZ is a transitional region 10–100 km wide between open sea and



**Figure 1.** Time lapse photographs spaced 6 minutes apart of a wave-induced breakup event of saline ice in the St. Lawrence Estuary. Courtesy, Dr Danny Dumont, 2015.

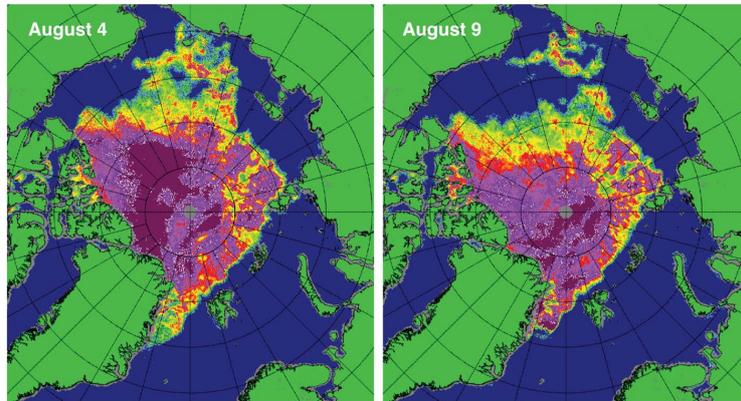
the interior pack ice described above. Of greatest importance to the present work is the observation that the penetrating waves are frequently destructive — they are the primary determinant of floe size in the MIZ because they induce flexural bending in the floes which can cause them to fracture, and they may also pummel ice so efficiently that a viscous slurry is created near the ice edge. Also significant is the well-known fact that waves diminish in amplitude as they progress further into the MIZ (see, e.g. [11], first published in 1919), due to both dissipation through turbulence and other processes, and the redistribution of energy by scattering. As a result, the floe size distribution (FSD) changes from near the ice edge where the waves are most intense and the most breakup occurs, to farther in where wave amplitude is less so the capability to destroy local floes is diminished. Scattering, although a conservative process over the entire ocean/ice continuum, causes attenuation and a reorganisation of angular spread in the directional wave spectrum towards a more isotropic wave field.

### 1.3. Ice breakup

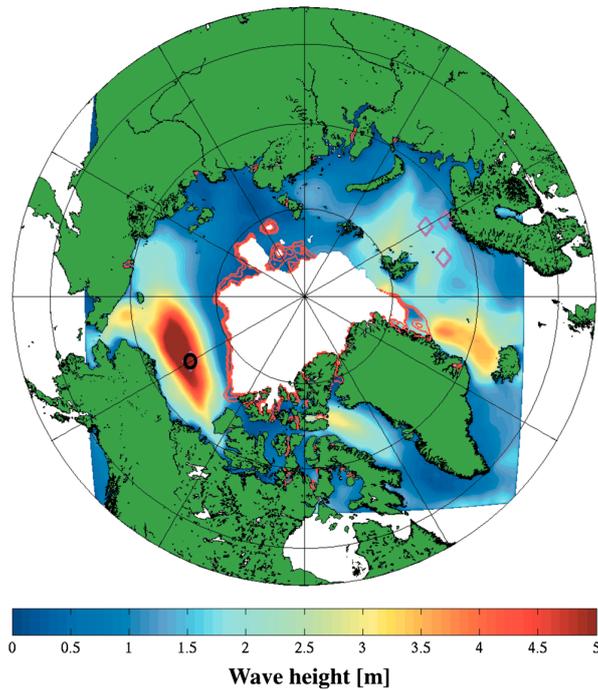
Ice breakup is discussed qualitatively in [12], including an illustration of the process. A sequential pair of recent photographs is also provided in Figure 1.

Recall from §1.1 that the incidence and severity of winds and waves is trending upwards [6], while the extent of Arctic sea ice is reducing; in fact each summer the ice now metamorphoses into a MIZ. Contrariwise Antarctic sea ice appears to be relatively stable or even growing in extent. Break up of ice floes has always occurred but, with global warming, it has increased in significance and, along with this, it has become of greater importance at large spatial scales because of its pronounced effect on FSD [13]. Breakup allows the floes to move about more freely, as seen in the before-and-after pictures of an Arctic storm shown in Figure 2. In September that same year, hindcast WAVEWATCH III<sup>®</sup> wave modelling with validation illustrates the additional fetches that can lead to very high waves and, inevitably, to further destruction of the sea ice. See Figure 3.

Breakup has another major effect that tends to get forgotten, which arises



**Figure 2.** Microwave radiometer concentration maps highlight the rapid loss of sea ice in the western Arctic (northwest of Alaska) during the strong Arctic storm of 2012. Magenta and purple colours indicate ice concentration near 100%; yellow, green, and pale blue indicate 60% to 20% ice concentration. Over three consecutive days (August 7, 8, and 9), sea ice extent dropped by nearly 200,000 square kilometres. Image: National Snow and Ice Data Center, courtesy IUP Bremen.



**Figure 3.** Example wave model hindcast during a storm in September 2012. The map is centred on the North Pole, and the mooring location is indicated by the black circle north of Alaska. The colour scale indicates significant wave height from 0 to 5 m. Reproduced from [4].

from the enhanced capacity of smaller ice floes to move laterally as a result of winds, currents and the ocean waves themselves; the latter as a result of radiation stress and Stokes drift. Under on-ice winds, the ice floes cluster together as more floes are crammed into a smaller area as the ice field compacts. When the wind is off-ice, concentration decreases and regions of open water appear either as leads or polynyas<sup>1</sup>. This is where the Arctic differs from the Antarctic. In the Arctic the major reduction in ice extent has occurred during the summer as a result of ice-albedo temperature feedback where the surface of the ice melts, spatially-averaged albedo decreases as melted ice is darker than solid snow-covered ice, and so more solar radiation enters the ice to encourage even more melting. Breakup by waves nourishes this process, because it also decreases the average albedo and helps the melting process as the water between the smaller ice floes is heated and melts ice laterally leading to intensified decay [13]. Antarctic ice on the other hand, only exists to any great extent during the winter months; it is the world's largest MIZ. So, when ocean waves break up the growing sea ice and the resulting floes march northwards driven by winds blowing off the Antarctic ice cap, leads and polynyas quickly freeze over to create more ice, i.e. ice breakup in the Southern Ocean encourages ice growth rather than discouraging it. Incidentally, other mechanisms to create more sea ice act in concert; contemporary research topics that are provoking heated debate.

The breakup of sea ice and the ensuing creation of a FSD is a pivotal component to ice/ocean models, climate models and effective global forecasting that is only now receiving major interest.

## 2. A simple beam fracture model

Consider a monochromatic train of 1-m-amplitude open ocean surface gravity waves propagating in the horizontal  $x$ -direction into an homogeneous thin (Euler-Bernoulli) beam of sea ice of thickness  $h$ , which we shall assume behaves elastically with its Young's modulus  $E$  constant for now. At the water-ice boundary the wave number will change abruptly and some reflection will occur because the boundary condition alters from an open water one to one that expresses the ice properties. Within the ice beam a corresponding train of so-called flexural-gravity waves of amplitude  $A_i \neq 1$  m, wave number  $k_i$  and radial frequency  $\omega$  will be produced. Define the vertical  $z$ -axis to point downwards from its origin located on the neutral plane, which is  $\delta$  from the upper ice surface and  $(h - \delta)$  from the bottom ice surface in general. For such a beam experiencing simple bending, relationships exist between longitudinal stress and strain,  $\sigma_{xx}$  and  $\varepsilon_{xx}$ , respectively, and the vertical bending displacement  $\eta$ , namely

$$\varepsilon_{xx} = -z\eta_{,xx}, \quad \sigma_{xx} = E\varepsilon_{xx} = -zE\eta_{,xx}. \quad (1)$$

Because  $E$  doesn't depend on  $z$ ,  $\delta = h/2$ , so on the top and bottom of the beam,

$$\varepsilon_{xx} = \pm \frac{h}{2} \eta_{,xx}, \quad \sigma_{xx} = E\varepsilon = \pm \frac{h}{2} E\eta_{,xx}. \quad (2)$$

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<sup>1</sup>a polynya is a lake in the ice cover

When the incoming wave field is no longer monochromatic but is instead composed of an aggregation of waves of the form  $\eta_i = A_i \cos(k_i x - \omega t)$  defined by a spectral density function  $S(\omega)$ ,  $A_i = A_i(\omega)$  is the amplitude response at each frequency of the ice beam to forcing from a wave of unit amplitude in the neighbouring open sea. Then, the mean square stress  $\langle \sigma^2 \rangle$  and mean square strain  $\langle \varepsilon^2 \rangle$  arise from

$$\langle \sigma^2 \rangle = E^2 \langle \varepsilon^2 \rangle, \quad \langle \varepsilon^2 \rangle = \int_0^\infty S(\omega) \mathcal{E}^2(\omega) d\omega, \quad \mathcal{E}(\omega) = \frac{1}{2} h k_i^2 A_i(\omega). \quad (3)$$

Expressions (3) don't account for nonlinear interactions between frequencies, which could potentially be important approaching an ice breakage event. Furthermore, brittle failure of the ice is being assumed so that a linear stress-strain law applies right up to the point where the ice breaks. Concordant with the definition of significant wave height, [14] defines the significant strain amplitude to be  $\mathcal{E}_s = 2\sqrt{\langle \varepsilon^2 \rangle}$ , i.e. two standard deviations in strain. Then the probability of the maximum strain from a passing flexural-gravity wave  $\mathcal{E}$  exceeding a breaking strain  $\varepsilon_c$  is  $\mathbb{P}_\varepsilon = \mathbb{P}(\mathcal{E} > \varepsilon_c) = \exp(-\varepsilon_c^2/2\langle \varepsilon^2 \rangle) = \exp(-2\varepsilon_c^2/\mathcal{E}_s^2)$ , which can be used in an ice/ocean model to evolve FSD [15]. However, we will not pursue this here. Naturally, a comparable formulation can be used for stress, with flexural strength  $\sigma_c$  replacing  $\varepsilon_c$ .

In theory, equation (3) is straightforward to apply, as we can calculate how the wave number changes and how much wave energy is reflected at a water-ice boundary, subject to knowing the physical properties of the uniform ice beam (using, e.g. [16] at normal incidence). Nevertheless, an irremediable shortcoming is that the value of  $\varepsilon_c$  is not known at all well. Only a very small data set is available for the wave-induced breaking strain of sea ice *in situ*, where values of approximately  $(5 \pm 3) \times 10^{-5}$  have been found. Without derogating caveats expressed in §1.2 about the labyrinthine material complexity of sea ice and our lack of hard data on  $\varepsilon_c$ , in the next section we attempt to find a way forward by exploiting empirical information about flexural strength  $\sigma_c$ . Later, we shall return to the issue of the substantial temperature gradient that exists between the top and bottom surfaces of sheets of sea ice, and the accompanying salinity profile.

## 2.1. Flexural strength

An up-to-date synopsis of our current knowledge about the engineering properties of sea ice does exist [17], which includes helpful information on flexural strength, Young's modulus and Poisson's ratio. Flexural strength has been measured many times due to its importance and use in ice engineering problems, granting that most data relate to first-year ice rather than multi-year ice as the latter experiments are arduous to conduct. To proceed, we will follow [14] and utilize experimental findings on flexural strength  $\sigma_c$ , even though we recognize that ice fracture is considerably more complicated than this and that, because of the nonuniform stress fields noted earlier, flexural strength is actually not a basic material property but an indicial strength.

Based on 939 flexural strength measurements, the empirical expression

$$\sigma_c = \sigma_0 \exp(-5.88\sqrt{v_b}), \quad \sigma_0 = 1.76 \text{ MPa}, \quad (4)$$

[18], relates flexural strength  $\sigma_c$  to brine volume fraction  $v_b$ . This shows a monotonic decrease from  $\sigma_0$  as  $v_b$  increases. Brine volume can be found from ice temperature and salinity [19].

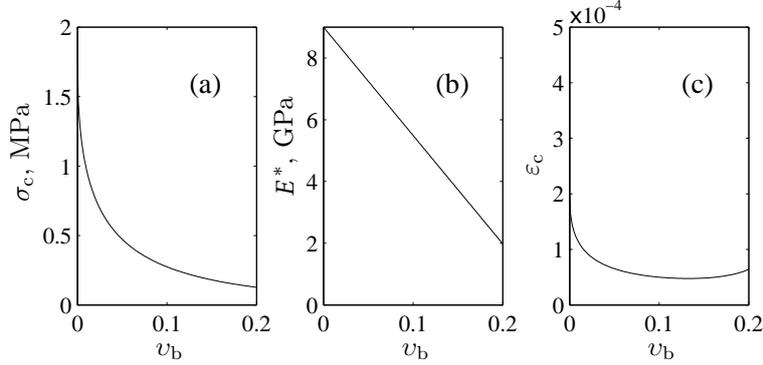
Flexural strength tests are invariably analyzed using Euler-Bernoulli beam theory, in which  $\sigma_{xx} = E\varepsilon_{xx}$  is used to relate stress and strain. Recall  $E$  denotes the Young's modulus, assumed constant for now. Unfortunately, as well as instantaneous elasticity  $\varepsilon^i$ , the constitutive relation for ice is rate-dependent [18]. It involves a delayed elastic (anelastic) response (primary, recoverable creep),  $\varepsilon^d$ , viscous strain (secondary creep) and strain due to cracking (tertiary creep). For modest strains and timescales,  $\varepsilon^i$  and  $\varepsilon^d$  are of particular importance, as both are impermanent although recovery from delayed elasticity is not instantaneous. In the course of a standard flexural strength test and during the recurring cyclic flexure imparted by ocean surface gravity waves, it is expected that the sea ice will experience stress levels and rates such that the total recoverable strain approximates  $\varepsilon^i + \varepsilon^d$ . This suggests a variation on the instantaneous elastic  $E$  that allows delayed elasticity to act, which we denote by  $E^*$ .  $E^*$  is usually called the effective modulus or the strain modulus.

$E$  itself can only be measured dynamically, e.g. ultrasonically with small isolated samples or *in situ* by measuring the propagation of high frequency elastic waves, while  $E^* < E$  arises from assuming an elastic constitutive relation in an experiment where some recoverable primary creep occurs. Moreover, as with flexural strength, brine volume affects both  $E$  and  $E^*$  such that empirically  $E = E_0(1 - 3.51v_b)$ , where  $E_0 = 10 \text{ GPa}$  [17]. Regrettably, whilst increased  $v_b$  will undoubtedly cause a reduction in  $E^*$  as well, data are too scattered for an equivalent empirical relationship for  $E^*(v_b)$  to be expressed categorically. For "average"  $v_b = 0.05\text{--}0.1$  [19], the formula suggests  $E$  will reduce to between  $\sim 6\text{--}8 \text{ GPa}$  and a similar reduction is conjectured for  $E^*$ .

More challenging to determine is the effect of anelasticity on reducing  $E$  to  $E^*$ , caused by the ice relaxing during cyclical loading because of power-law primary creep with no microcracking. Here rate of loading is important. Few data can help us but Figure 4 of [20] shows model predictions for the effective modulus at four loading frequencies that include those associated with surface gravity wave periods, i.e. 0.01–1 Hz, and, incidentally, the reduction in  $E$  due to total porosity, i.e. air plus brine. The latter effects are comparable in magnitude to the reductions in  $E$  given above; the effect of rate is about 0.5 GPa as wave period is changed from 1 s to 10 s, and about 1 GPa from 10 s to 100 s. We therefore consider a reduction of 1 GPa is reasonable, so finally write

$$\sigma_c = E^*\varepsilon_c, \quad E^* = (9 - 35.1v_b) \text{ GPa}, \quad (5)$$

reminding the reader that this is an empirically derived quantity that is, at best, a guesstimate of reality that doesn't reflect the remarkable variability of sea ice in nature.



**Figure 4.** Behaviour of the flexural strength (a), and our models for the effective modulus (b) and the breaking strain (c) with the brine volume fraction  $v_b$ .

The effective modulus and breaking strain given by equation (5) are plotted as functions of brine volume fraction in parts (b) and (c) of Figure 4. The same value of  $E^*$  must be used in the boundary condition that represents the floating ice of course, where it enters via the beam's flexural rigidity per unit width,  $D = E^*h^3/12$ . Apropos breaking strain,  $\varepsilon_c$  has a minimum value of approximately  $4.8 \times 10^{-5}$  when  $v_b = 0.15$  where  $E^* = 3.8$  GPa, which is fortuitously close to the  $(5 \pm 3) \times 10^{-5}$  measured *in situ*. The value of  $\varepsilon_c$  is roughly constant for  $v_b \in [0.1, 0.2]$  but increases at higher and lower brine volumes — the less porous ice is predictably stronger, while the more porous ice is more compliant so will be able to sustain more bending before breaking. Ice fatigue arising from cumulative damage due to cyclic loading by the waves is also a possibility [21], whether of the high-cycle type associated with elastic behaviour and growth of microscopic cracks that eventually reach a critical size for fracture, or low cycle fatigue where the stress is sufficient for plastic deformation. Data around this are too equivocal to be included.

### 3. A better model of ice failure

As foreshadowed earlier in the text, we now investigate whether the temperature and salinity gradients that exist across the ice thickness affect ice failure. To do this, it is necessary to allow the elastic modulus to vary in some prescribed way through the ice thickness, which leads to an amended flexural rigidity term

$$D = \int_{-\delta}^{h-\delta} z^2 E(z) dz, \quad (6)$$

per unit width for a beam [22], recollecting that  $h$  is ice thickness and  $\delta$  is the distance between the neutral plane and the upper surface, which is defined by

$$\int_{-\delta}^{h-\delta} zE(z)dz = 0. \quad (7)$$

Accordingly, the value of  $D$  depends on the profile of  $E(z)$  through the thickness, and the neutral plane is only halfway between the upper and lower surfaces when  $E$  is constant. It remains to specify  $E(z)$ , of course, which is achieved from temperature and salinity measurements through the ice thickness, [19] and equation (5). [22] provides a general, physically-plausible, algebraically-convenient form for  $E(z)$ , as follows

$$E(z) = E^* \left( 1 - (1 - \alpha) \left( \frac{z}{h} + \frac{\delta}{h} \right)^n \right), \quad (8)$$

where  $E^*$  is now the effective modulus at the upper ice surface where  $v_b$  is relatively small unless the ice surface is permeated with brine; see Figure 4. Parameter  $\alpha \in [0, 1]$  expresses the difference between the elastic modulus at the upper and lower surfaces, which are the same when  $\alpha = 1$ , while  $n$  encapsulates the gradient with  $n = 1$  designating a linear variation. The neutral plane is now located at

$$\delta = \frac{h}{2} \frac{(n + 2\alpha)(n + 1)}{(n + \alpha)(n + 2)}, \quad (9)$$

from the upper surface, which gives, finally,

$$D = \frac{E^* h^3}{12} \left( \frac{4(n + 2)^2(n + \alpha)(n + 3\alpha) - 3(n + 1)(n + 3)(n + 2\alpha)^2}{(n + 2)^2(n + 3)(n + \alpha)} \right), \quad (10)$$

so that when  $\alpha = 1$ ,  $D = E^* h^3 / 12$ , as expected. A heterogeneous profile through the sea ice plate turns out to be very significant. For  $\alpha = 0.2, n = 2$ ,  $D$  is reduced by a factor 0.61; for  $\alpha = 0.2, n = 0.5$ ,  $D$  is reduced by a factor 0.42. Both are reasonable profiles that one would expect to measure through the sea ice sheet, where the bottom of sea ice has a low solid fraction that is infiltrated by brine. It is imperative to appreciate that  $D$  enters our model of wave-induced breakup both in the calculation of the flexural-gravity wave profile in the beam, i.e. amplitude and wave number, and through the flexural stress  $\sigma_{xx} = -zE(z)\eta_{,xx}$ . On the upper surface  $z = -\delta$ ,  $E = E^*$  and  $\sigma^U = \delta E^* \eta_{,xx}$ , while on the lower surface  $z = (h - \delta)$ ,  $E = \alpha E^*$  and  $\sigma^L = -\alpha(h - \delta)E^* \eta_{,xx}$ , so

$$\left| \frac{\sigma^L}{\sigma^U} \right| = \alpha(h/\delta - 1) = \alpha \left( 2 \frac{(n + \alpha)(n + 2)}{(n + 2\alpha)(n + 1)} - 1 \right) \leq 1, \quad (11)$$

with equality satisfied only when  $\alpha = 1$  when  $\delta = h/2$ , from expression (9). Consequently, with a realistic sea ice profile for  $E(z)$ , the neutral plane is displaced towards the upper surface where the largest longitudinal stress occurs. Unfortunately, this does not necessarily mean that the sea ice will fracture at

its upper surface, as this will depend on the relative magnitude of  $\sigma_{xx}$  and the flexural strength  $\sigma_c$ , which is a function of brine volume as seen in part (a) of Figure 4. Moreover, even with the Euler-Bernoulli plane-section hypothesis, because  $E = E(z)$  the distribution of stresses is not linear in  $z$ . Near the underside of the sea ice beam, the solid fraction is low and the brine volume is usually high ( $> 0.3$ ), potentially 5 times or more larger than the brine volume at the top surface. Concomitantly, this suggests with regard to  $\sigma_c$ , that the ice at the surface is about 5 times stronger than the ice at the bottom. Assuming a linear change in  $E^*$  from top to bottom, i.e.  $n = 1$ , we can roughly reproduce this disparity by choosing  $\alpha \approx 0.1$ , acknowledging from equation (5) that a linear variation in  $E^*$  suggests a linear variation in  $v_b$ . While this is unphysical for sea ice, choosing nonlinear profiles for  $E^*$ , i.e.  $n \neq 1$ , doesn't change the outcome greatly, and leads us to the rather unhelpful conclusion that we cannot conclusively argue that the ice beam will fracture first at the top or at the bottom.

Rather than solving the periodic case of a wave train interacting with a floating ice beam, [23] considers the effect of a static sinusoidal lateral load. Two solutions are provided for heterogeneous beams, one for brittle failure at the underside and the other when the ice is allowed to creep with a power exponent of 3, i.e. following Glen's flow law. While innovative, there is no evidence that sea ice is susceptible to this kind of creep behaviour at the strain rates representative of ocean waves.

#### 4. A floating ice plate

Ice floes are not actually floating beams, and there will be many wave-ice interaction problems that require the sea ice to be represented as a plate rather than a beam, notably obliquely incident waves and circular ice floes. Kirchhoff-Love theory extends the Euler-Bernoulli beam, allowing the stresses and deformations in thin plates subjected to forces and moments to be found. Designating the  $y$ -direction perpendicular to  $x$  and  $z$  in the neutral plane and denoting Poisson's ratio by  $\nu \approx 0.3$  for most sea ice, this theory replaces equation (1) with

$$(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = -\frac{zE(z)}{1-\nu^2} (\eta_{,xx} + \nu\eta_{,yy}, \eta_{,yy} + \nu\eta_{,xx}, (1-\nu)\eta_{,xy}). \quad (12)$$

Flexural rigidity must be revised from expression (6), to be

$$D = \frac{1}{1-\nu^2} \int_{-\delta}^{h-\delta} z^2 E(z) dz, \quad (13)$$

so, as required,  $D = Eh^3/(12(1-\nu^2))$  if  $E$  is constant. Using (13) to define  $D$ , we can derive the customary plate equation for wave-ice interactions by employing the Kirchhoff-Love assumptions subject to (7). In sum, the beam and the plate can be treated similarly, as long as  $D$  includes the  $1/(1-\nu^2) \approx 1.1$  factor as in definition (13).

## 5. Summary and conclusions

On the face of it, the model of ice breakup we have considered in this paper is not new, as wave-induced cyclical bending in its simplest form just compares the largest extensional strain in flexure with what we know about the strain magnitude required to break the ice beam or plate. Unfortunately, we don't know a great deal about the latter, as few observations have occurred that have actually measured the strain during a wave breakup event. As a result, our focus here has actually been somewhat different as, faced with that serious dearth of data, we have (i) sought to utilize other expedient datasets, in particular flexural strength measurements; and (ii) investigated the effect of a much more plausible model of sea ice that accommodates a major feature of its physical properties, namely the variation of its material coefficients through its thickness that arises because of pervasive temperature and salinity gradients. Inevitably, this extra complexity causes challenges, primarily in regard to the nonlinear distribution of stresses created, which makes it difficult to know where the beam or plate will actually break (or creep if an inelastic constitutive relation is used), but it also tells us that we have to decrease our integrated flexural rigidity term in the equation of motion quite significantly over and above the reduction suggested by arguments about anelastic effects and strain rates. These are useful outcomes that will need to be fed into ice/ocean models that assimilate the widespread upward trends in ocean wave intensity that global warming has fuelled.

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