Extremal behavior of a coupled continuous time random walk

Rina Schumer[1,*]

Division of Hydrologic Sciences, Desert Research Institute, Reno, NV 89512

Boris Baeumer

Department of Mathematics & Statistics, University of Otago, Dunedin, New Zealand

Mark M. Meerschaert[2]

Department of Statistics & Probability, Michigan State University, East Lansing, MI 48824

Abstract

Coupled continuous time random walks (CTRW) model normal and anomalous diffusion of random walkers by taking the sum of random jump lengths dependent on the random waiting times immediately preceding each jump. They are used to simulate diffusion-like processes in econophysics such as stock market fluctuations, where jumps represent financial market microstructure like log-returns. In this and many other applications, the magnitude of the largest observations (e.g. a stock market crash) is of considerable importance in quantifying risk. We use a stochastic process called a coupled continuous time random maxima (CTRM) to determine the density govern-

[*]Corresponding Author

Email addresses: rina@dri.edu (Rina Schumer), bbaeumer@maths.otago.ac.nz (Boris Baeumer), mcubed@stt.msu.edu (Mark M. Meerschaert)

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ing the maximum jump length of a particle undergoing a CTRW. CTRM are similar to continuous time random walks but track maxima instead of sums. The many ways in which observations can depend on waiting times can produce an equally large number of CTRM governing density shapes. We compare densities governing coupled CTRM with their uncoupled counterparts for three simple observation/wait dependence structures.
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1. Introduction

Random walks have long been used to model behavior of random diffusive-type processes in finance and econophysics [1]. Price fluctuations are likened to particles undergoing random, though well-behaved, motions and long term market behavior is represented by sums of particle jumps. Continuous time random walks (CTRW) generalize classical models by permitting particle jumps and waiting times between jumps to have arbitrary probability densities and allowing representation of anomalous (super- or sub-diffusive) behavior. Thus, CTRW have been used to reproduce the behavior of price fluctuations in financial markets [2, 3, 4, 5, 6]. Recently, coupled CTRW, in which size of an observation is dependent on the length of the preceding waiting period, have been used to describe movement of stock or share prices [7] [8], electricity markets [9], currency exchange [10],and other types of financial data behavior [11]. If the practitioner is interested in the largest price shock expected in a given interval rather than long term evolution of prices, then extreme value models are appropriate tools. Extreme events in financial time series have been characterized using the mean first-passage time and mean exit time of CTRW [12]. We seek the density governing the maximum price fluctuation.

Classical extreme value models describe the density governing the largest of independent and identically distributed (iid) random observations with arbitrary probability density, independent of fixed-length or exponentially distributed interarrivals. If observations are particle jumps, then these models describe recurrence of the jumps of a particle undergoing a random walk. A generalization to accommodate an arbitrary waiting time between observations was recently accomplished in [13]. In this work, we extend that model to allow the distribution of an observation to depend on the waiting time before that observation. The purpose of this generalization is to estimate recurrence intervals for random phenomena that are well modeled by coupled CTRWs. Continuous time random maxima (CTRM) models are stochastic processes that track the largest observation in a series events separated by random waiting times. They can be used to forecast the largest particle observation or price jump with time.

2. Coupled continuous time random maxima

Our goal is to estimate the density governing the maximum jump length that a particle undergoing a CTRW with iid jumps dependent on the iid waiting times that separate them. We call the space-time stochastic process $\{M_t\}_{t\geq 0}$ a *coupled continuous time random maxima (CTRM)* because its form is similar to a CTRW but it tracks jump maxima instead of sums.

Let J_n = iid duration of interarrivals with density $\psi(t)$, X_n = iid event magnitudes,$T_n = J_1 + J_2 + \cdots + J_n$ = time of the n^{th} event, and $N_t = \max\{n : T_n \leq t\}$ = number of events by time t. To be complete, we also set $T_0 = 0$ and $M_0 = -\infty$. Further let $M(n) = \max(X_1, X_2, \cdots, X_n)$ = maximum of the first n observed events and $M_t = M(N_t) = \max(X_1, X_2, \cdots, X_{N_t})$ = maximum observed event by time t. The joint density of the coupled waiting time/ event magnitudes is denoted $(X_i, J_i) \sim f(x,t)$. The waiting time density is the time marginal of the transition density $\psi(t) = \int f(x,t)\,dx$. Using the inverse relationship between the time until the nth event and the number of events by time t $\{N_t \geq n\} = \{T_n \leq t\}$ and the fact that the time between events n and $n+1$ is the $n+1^{st}$ wait J_{n+1} we find

$$\begin{aligned}
&P(M(N_t) \leq x) \\
&= \sum_{n=0}^{\infty} P(M(n) \leq x, N_t = n) \\
&= \sum_{n=0}^{\infty} [P(M(n) \leq x, N_t \geq n) - P(M(n) \leq x, N_t \geq n+1)] \\
&= \sum_{n=0}^{\infty} [P(M(n) \leq x, T_n \leq t) - P(M(n) \leq x, T_{n+1} \leq t)] \\
&= \sum_{n=0}^{\infty} [P(M(n) \leq x, T_n \leq t) - P(M(n) \leq x, T_n + J_{n+1} \leq t)] \\
&= \sum_{n=0}^{\infty} \Big[P(M(n) \leq x, T_n \leq t) \\
&\qquad\qquad - \int_0^{\infty} P(M(n) \leq x, T_n \leq t - \tau)\psi(\tau)\,d\tau\Big],
\end{aligned} \tag{1}$$

where the integral in the last line of (1) includes the probability of all possible waiting times between jumps n and $n+1$. Note that the distribution of $M(N_t)$ has an atom at $x = -\infty$, equal to the probability $P(N_t = 0) = P(J_1 > t)$

that the first observation has not yet occurred by time t. Let $F(x,t) = \int_{-\infty}^{x} f(u,t)\,du$ and define the CDF-Laplace (C-L) transform of f to be

$$\bar{f}(x,s) = \int_0^\infty e^{-st} F(x,t)\,dt = E[e^{-sJ} I(X \le x)],$$

which at $x = \infty$ is the Laplace transform of ψ and at $s = 0$ is the CDF of X. While the density of a sum is the convolution (product in transform space) of the individual summand densities, the cumulative distribution function (cdf) governing the maxima of a set of iid random variables is the product of the individual cdfs. This is reflected by the fact that the density of $(M(n), T_n)$ has C-L transform

$$E[e^{-sT_n} I(M_n \le x)] = E\left[\prod_{i=1}^{n} e^{-sT_i} I(M_i \le x)\right] = \prod_{i=1}^{n} E[e^{-sT_i} I(M_i \le x)] = \bar{f}(x,s)^n$$

which is also true for $n = 0$.

Next, integrate by parts to get

$$\begin{aligned}
\int_0^\infty e^{-st} P(M(n) \le x, T_n \le t)\,dt &= E\left[\int_0^\infty e^{-st} I(M(n) \le x, T_n \le t)\,dt\right] \\
&= E\left[s^{-1} \int_0^\infty e^{-st} I(M(n) \le x | T_n = t) \psi(t)\,dt\right] \\
&= s^{-1} \int_0^\infty E\left[I(M(n) \le x) e^{-sT_n} | T_n = t\right] \psi(t)\,dt \\
&= s^{-1} \int_0^\infty E\left[I(M(n) \le x) e^{-sT_n} | T_n = t\right] \psi(t)\,dt \\
&= s^{-1} E\left[I(M(n) \le x) e^{-sT_n}\right] \\
&= s^{-1} \bar{f}(x,s)^n
\end{aligned} \tag{2}$$

by conditioning on $T_n = t$. Let $M(x,t) = P(M(N_t) \le x)$ and take Laplace transforms in (1) to get

$$\begin{aligned}\tilde{M}(x,s) &= \sum_{n=0}^{\infty}\left[s^{-1}\bar{f}(x,s)^n - s^{-1}\bar{f}(x,s)^n\tilde{\psi}(s)\right] \\ &= \frac{1-\tilde{\psi}(s)}{s}\frac{1}{1-\bar{f}(x,s)},\end{aligned} \tag{3}$$

since $\sum_{i=0}^{\infty} r^i = 1/(1-r)$, provided that $|r| < 1$. Multiplying both sides of (3) by $1-\bar{f}(x,s)$ and inverting the Laplace transform yields that

$$M(x,t) - \int_0^t F(x,t-\tau)M(x,\tau)\,d\tau = \int_t^{\infty}\psi(\tau)\,d\tau, \tag{4}$$

and therefore for each x, $M(x,t)$ is the solution of a Volterra integral equation (4) of the second kind [14].

This master equation for coupled CTRM reduces to the master equation for uncoupled CTRM when event magnitudes with distribution function $F_e(x)$ and interarrivals are uncorrelated so that $\bar{F}(x,s) = F_e(x)\tilde{\psi}(s)$ [13]:

$$\mathcal{L}[P(M(N_t) \le x)] = \frac{1-\tilde{\psi}(s)}{s}\frac{1}{1-\tilde{F}_e(x)\psi(s)} \tag{5}$$

or

$$M(x,t) - F_e(x)\int_0^t \psi(t-\tau)M(x,\tau)\,d\tau = \int_t^{\infty}\psi(\tau)\,d\tau. \tag{6}$$

3. Examples

Here we demonstrate the effect of a coupled wait/observation density on the probability distribution function (pdf) of the maxima predicted by the CTRM model, where observations may represent financial returns or actuarial claims. In general, the densities governing the maximum value (Equation 4) do not exist in closed form and have to be computed numerically. Here, we use a basic stepping method to generate densities (Appendix); for more refined numerical methods, see for example [15].

3.1. Exponential waiting times and jump distributions

Example 1: Here we assign an exponential distribution with mean $\lambda_t = 1/10$ to the waiting times J_i between observations. We also assign the same distribution, exponential with mean $1/\lambda_x = 10$, to the observations Y_i that occur at the end of each waiting time. We compare the pdf of the maximum observation in two very different cases: (a) observations Y_i are independent of waiting times J_i; and (b) observations are coupled to length of the waiting times by $Y_i = J_i$. Since J_i are exponential, N_t has a Poisson distribution with mean $\lambda_t t$ in this example. Then case (a) represents the maximum observation, $M(N_t)$, for a compound Poisson process. Since Y_i has light tails, the pdf of $M(N_t)$ is governed at late time by a Gumbel density [16]. In fact, for $x \geq 0$ we have

$$\begin{aligned}
P(M(N_t) \leq x) &= E(P(M(N_t) \leq x | N_t)) \\
&= \sum_{n=0}^{\infty} P(M(N_t) \leq x | N_t = n) P(N_t = n) \\
&= \sum_{n=0}^{\infty} P(M_n \leq x) P(N_t = n) \\
&= \sum_{n=0}^{\infty} (1 - e^{-\lambda_x x})^n e^{-\lambda_t t} \frac{(\lambda_t t)^n}{n!} \\
&= e^{-\lambda_t t} \sum_{n=0}^{\infty} \frac{[(\lambda_t t)(1 - e^{-\lambda_x x})]^n}{n!} \\
&= e^{-\lambda_t t} e^{(\lambda_t t)(1 - e^{-\lambda_x x})} \\
&= e^{-\lambda_t t e^{-\lambda_x x}}
\end{aligned}$$

which is exactly the Gumbel formula. If $x \leq 0$ then we have $P(M(N_t) \leq x) = e^{-\lambda_t t}$ since $M(N_t) = -\infty$ when $N_t = 0$, which happens with probability

$P(J_i > t) = e^{-\lambda_t t}$ in this example. The only difference between the Gumbel and the pdf of $M(N_t)$ in this example is that the probability assigned to $x < 0$ under the Gumbel pdf is concentrated at $x = -\infty$ in the pdf of $M(N_t)$. Since this probability tends to zero as t increases, the asymptotic distribution of $M(N_t)$ is exactly Gumbel.

Figure 1 shows that pdf of the largest observation at $t = 100$. The coupled pdf has thinner tails and higher peak than the uncoupled model, reflecting the decreased probability of large or small observations when event magnitude is tied to the preceding waiting time. The atom $P(J_i > 100) = e^{-10}$ is negligible in this case.

Example 2: Now suppose that the waiting times J_i between observations have a standard Pareto distribution with tail parameter β=0.7: $P(J_i > t) = (1+t)^{-0.7}$ for $t > 0$. As in Example 1, we compare pdfs that arise from the CTRM model if: (a) observations Y_i have Pareto (β=0.7) distribution independent from that of the waiting times; and (b) observations Y_i are equal to the preceding wait time J_i. Figure 2 compares the pdf for each case at time $t = 100$. Both densities have a peak at zero. The density for the uncoupled CTRM decreases monotonically from the peak at zero, following the decay of the observation density. In comparison, the coupled CTRM density is bimodal.

Example 3: The evolution of LIFFE bond future prices was modeled using a CTRW with coupling $Y_i = J_i^{1/2} Z_i$ where Z_i are standard normal random variables independent of J_i [7], implying that the variance of an observation is drawn from a mean zero Gaussian distribution with variance equal to the preceding waiting time. This coupling, originally proposed by [17], dictates that large observations tend to follow large waiting times and also bounds the rate Y_i/J_i at which prices can change. We compare uncoupled and coupled CTRM with Pareto waiting time distribution $P(J > t) = (1 + t)^{-0.7}$. In both cases, the probability of encountering a maxima less than zero decreases rapidly if more than one observation occurs by $t = 100$. Thus, both densities are concentrated on the positive line. The density for an uncoupled CTRM with standard Gaussian observations (dotted line, Figure 3) is near-symmetric with a sharp peak. In contrast, the density of $M(N_t)$ for the coupled CTRM is positively skewed because small observations can follow waits of any magnitude, while large observations can follow only long waiting times.

4. Discussion

Coupling of observation length and waiting time densities in a CTRM significantly alters the density of the maximum observation size with time. The framework we develop here can be used to describe the maximum observation sized expected from a coupled CTRW when waiting time and joint waiting/observation densities are exactly known. In the long time limit, dependence of observations Y_i on waiting times J_i can disappear [7]. Just as tails of the waiting time and observation densities determine max-stable density that will arise, dependence in the tails (i.e. of the largest events) determine whether extremal limit theorems for uncoupled random variables [18, 19, 20] or coupled [21] will apply. In a useful extension of theory of maxima for CTRW, the joint behavior of the largest observation and the long term behavior of the sum process (coined the max-sum processes with renewal stopping) is established by [21].

As a specific example of the many, diverse applications of CTRM distributions, we consider here the application to finance proposed in [7]. The observations Y_i are log-returns and J_i are the waiting times between trades, for a bond future. Then N_t is the number of trades by time $t > 0$, and $M(N_t)$ is the biggest log-return by time t. The CTRM pdf thus quantifies the risk (or opportunity) of a large price jump in the interval $(0, t)$. It is clear from Figure 3 that coupling can significantly increase this risk. For this reason, ignoring the coupling can lead to a seriously deficient strategy for assessing the risk of a large price change, which is in fact considerably greater than the uncoupled model suggests.

5. Conclusions

In many applications, magnitude of the largest events and estimates of their recurrence are central to risk analysis. Coupled CTRM are random walks in space-time that quantify likelihood of the largest observations in a given interval for processes whose evolution are well-represented by coupled CTRW.

Appendix: Numerical approximation of Volterra equations

In order to approximate the solution to (4), we approximated the convolution integral using Simpson's rule; i.e. for a given time step dt and $t \approx n\,dt$,

$$\int_0^t F(x,t-\tau)M(x,\tau)\,d\tau \approx dt\sum_{i=1}^{n-1} F(x,(n-i)\,dt)M(x,i\,dt) + \frac{dt}{2}F(x,n\,dt)M(x,0) + \frac{dt}{2}F(x,0)M(x,n\,dt). \tag{7}$$

Hence we step by approximating $M(x,(n+1)\,dt)$ with $M_{n+1}(x)$ satisfying

$$\begin{aligned} M_{n+1}(x) = & dt\sum_{i=1}^{n} F(x,(n-i)\,dt)M_i(x) \\ & + \frac{dt}{2}F(x,n\,dt)M_0(x) + \frac{dt}{2}F(x,0)M_{n+1}(x) + \int_{(n+1)dt}^{\infty} \psi(\tau)\,d\tau, \end{aligned} \tag{8}$$

or

$$M_{n+1}(x) = \left(dt\sum_{i=1}^{n} F(x,(n-i)\,dt)M_i(x) + \frac{dt}{2}F(x,n\,dt)M_0(x) + \int_{(n+1)dt}^{\infty} \psi(\tau)\,d\tau \right) \bigg/ \left(1 - \frac{dt}{2}F(x,0)\right). \tag{9}$$

The density of M with respect to x needed for the figures can then be obtained by (central) differencing in x.

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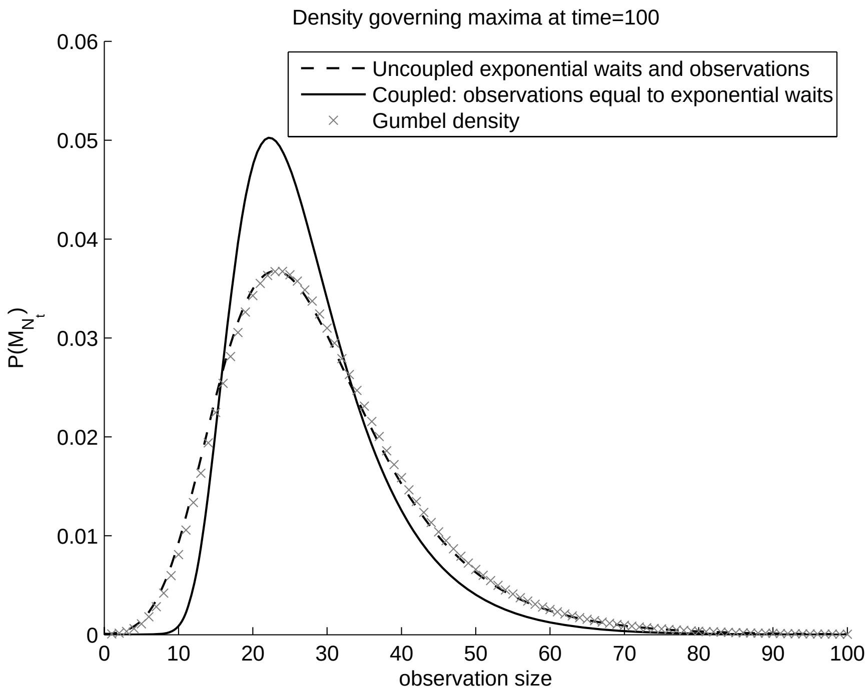

Figure 1: The density governing maximum observation size at time=100 for a CTRM with exponential observation length equal to the preceding independent exponential ($\lambda = 1/10$) waiting times (solid line) has thinner tails and a higher peak than the density for the analogous uncoupled model (dashed line).

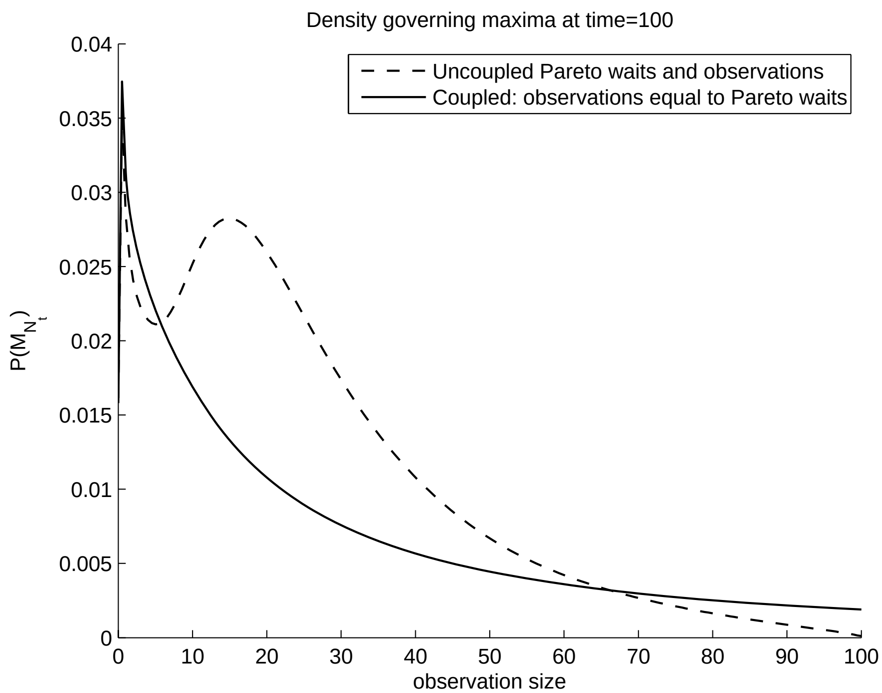

Figure 2: The density governing maximum observation size at time=100 for a CTRM with Pareto observations equal to the preceding independent Pareto ($\beta = 0.7$) waiting times (solid line) is bimodal and has a thinner tail than the density for the analogous uncoupled model (dashed line).

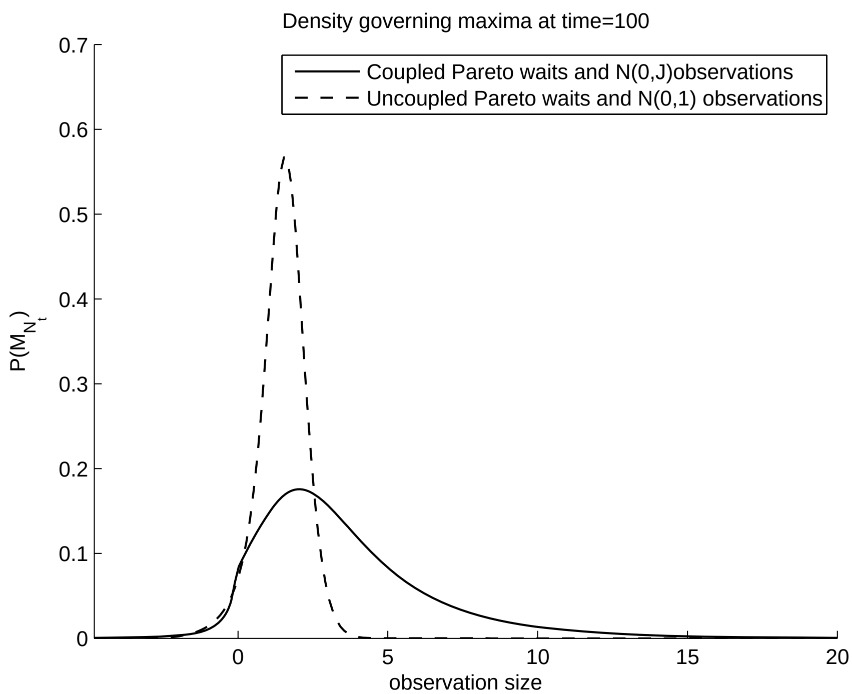

Figure 3: The density for an uncoupled CTRM with standard Gaussian observations vs. the density of $M(N_t)$ for a CTRM with coupling $Y_i = J_i^{\beta/\alpha} Z_i$, where Z_i are standard normal random variables independent of J_i.