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A Subordinated Kinematic Wave Equation for Heavy-Tailed Flow Responses from Heterogeneous Hillslopes

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Abstract.

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Abstract.

Analytical expressions of hillslope-scale subsurface stormflow discharge are currently restricted to hillslopes with homogeneous or mildly heterogeneous conductivity fields. In steep, straight hillslopes with uniform recharge these exhibit a classical piston flow response, which arises from an assemblage of impulses all moving at a constant velocity but with different starting locations. Heterogeneity within a hillslope soil creates variations in the downslope velocity of these impulses, which may lead to non-piston flow responses with either exponential or heavy (power-law) tails. The presence of heavy tails suggests that heterogeneity imparts a temporal memory on the motion of the impulses. Using this assumption, a subordinated kinematic wave equation is proposed for moderately to highly heterogeneous hillslopes. This equation convolves the piston response from a homogenous hillslope with a stable subordinator. The stable subordinator randomizes the time that impulses spend in motion and produces non-piston solutions with heavy tails. Through comparisons of synthetic data generated from numerical hillslope simulations with physically realistic parameters, this equation faithfully reproduces both early and late time characteristics of heavy-tailed flow responses from moderate to highly heterogenous hillslopes. A systematic evaluation of hillslope responses under different degrees of heterogeneity revealed a quantitative link between the statistical properties of the heterogeneous random fields and the parameters of the subordination framework. This suggests that the subor-

dinator can be parameterized with the physical measurement of hillslope properties.

1. Introduction

1.1. The closure problem

Modeling the flow of water through natural landscapes is necessary for accurate hydrologic prediction, as well as prediction of fluxes of pollutants, nutrients, pathogens and sediments. However, the heterogeneity present in these landscapes makes these predictions very difficult. It is practically infeasible to fully characterize the heterogeneity of the landscape properties at a sufficiently small scale to be commensurate with the well-known physical theories that describe hydrologic fluxes, such as Darcy's Law [*McDonnell et al.*, 2007]. A key challenge for advancing hydrology therefore is developing 'closure relations' [*Reggiani et al.*, 1999, 2000] – descriptions of hydrologic fluxes at some integral scale (such as a hillslope) in terms of parameters that are physically meaningful at those scales. These closure relations must then account for the effects of the spatial heterogeneity on the fluxes without the need to resolve the flows at smaller (sub-hillslope) scales explicitly [*Sivapalan*, 2005; *Beven*, 2006].

One such hydrologic flux is subsurface saturated lateral flow through hillslopes. This class of subsurface flow is typically characterized as occurring in steep terrain, where highly permeable surface soils are underlain by relatively impermeable bedrock or other aquitard. Recharge from the unsaturated zone reaches this layer and flows in a saturated layer under the influence of topographic and hydraulic gradients.

Mathematical formulations describing this flow at the scale of the hillslope have been developed for the case of no heterogeneity based on upscaling the point-scale laws that are assumed to describe the flux [*Beven*, 1981; *Brutsaert*, 1994; *Berne et al.*, 2005; *Pan-*

iconi et al., 2003; *Troch et al.*, 2004, 2003, 2002]. These models might be called 'zero-dimensional' since they predict behavior at the scale of the whole hillslope, and do not require that a differential equation be solved numerically to obtain the internal dynamics of the hillslope. Early work by *Beven* [1981] investigated the behavior for the restricted case of steep slopes and small saturated thickness where the topographic gradients dominate the flow. In these cases the solution behaves as a kinematic wave, in which flow velocity is everywhere proportional to the base gradient, and hydraulic gradients are relatively minor. This produces a piston-flow-like discharge response at the base of the hillslope. The later work of *Brutsaert* [1994] and others considered a wider range of circumstances, including more complex hillslope geometry. However, in all these cases the hydraulic properties of the soils are assumed to be uniform.

Presently, there are few similar 'zero-dimensional' hillslope-scale analytical models that account for the effect of spatial heterogeneity on transient hillslope-scale responses to recharge. In general, the large fluctuations in local water potential and flux associated with transient unconfined flow over a sloping base make it difficult to apply standard stochastic groundwater approaches. A perturbation type approach was adopted by *Dogrul et al.* [1998] to derive ensemble-average partial differential equations (PDEs) from the one-dimensional form of the modified Boussinesq equation, but found that these equations poorly represented the late-time water table profiles produced by flow simulations with relatively mild heterogeneous hydraulic conductivity fields ($\sigma_{LogK} \leq 1.2$). Subsequent work by Kavvas and colleagues has led to the development of the Watershed Environmental Hydrology (WEHY) model [*Kavvas et al.*, 2004; *Chen et al.*, 2004], which implements an upscaled conservation equation of two-dimensional, unconfined subsurface flow to account for random changes in hydraulic conductivity. Most recently, *Cayar and Kavvas* [2008] developed a mixed Eulerian-Lagrangian-Fokker-Planck equation based on the traditional Boussinesq equation to describe one-dimensional, unconfined subsurface flow through heterogeneous media. This approach was not applied to the case of a sloping base.

1.2. Effects of heterogeneity on hillslope-scale hydrologic response

Soils properties can vary a great deal over a range of scales, including the scale of a hillslope. For instance, *Mallants et al.* [1996] found that the saturated hydraulic conductivity of a well-drained colluvial subsoil in Belgium varied by up to 897% along a 31m transect. Recently, *Harman* [2007] (see also *Harman and Sivapalan* [2009a, b]) used a numerical model to study flow through heterogeneous hillslopes and found that the presence of heterogeneity produced response curves that fundamentally differ from those with uniform properties. Consequently, descriptions of the hillslope-scale response to recharge based on descriptions of flow through uniform hillslopes (such as the use of 'effective' parameters) have been found to be unlikely to give good predictions in cases with significant heterogeneity. *El-kadi and Brutsaert* [1985] used a simple numerical model to investigate whether an effective K value could be found that would reproduce the behavior of a modeled heterogeneous hillslope, but found (as *Dagan* [1982] did for the confined case) that the 'effective' value was not constant over time. *Binley et al.* [1989] obtained effective values of K for certain cases of simulated flow through heterogeneous hillslopes, but found that they varied between storm events, and were unable to find any relation between the effective values and the moments of the spatial distributions of K. *Fiori and Russo* [2007] used a full 3-D Richards equation solver to investigate the hillslope subsurface flow response and concluded that discharge from heterogeneous hillslopes is more sensitive to temporal variability in recharge inputs than homogeneous hillslopes due to the role of preferential pathways in transmitting recharge impulses.

In the modeling study of *Harman and Sivapalan* [2009b], the responses in relatively steep hillslopes ranged from classical piston flow in homogeneous soils, to non-piston responses characterized by either exponential or heavy-tailed flow recessions in heterogeneous soils. *Harman and Sivapalan* [2009b] suggested that the heterogeneous hillslope response could be explained by deviations in the velocity of flow above and below the constant velocity of the uniform case. It is well understood that the randomization of a parameter of a probability distribution (or in this case, the randomization of the velocity controlling the arrival time at the hillslope base) can give rise to distributions that do not resemble the parent distribution [e.g. *Ganti et al.*, 2009a, this issue; *Stark et al.*, 2009, this issue; *Hill et al.*, 2009, this issue]. This provides an intuitive (if not completely rigorous) explanation for the exponential and heavy-tailed responses observed by *Harman and Sivapalan* [2009b]. For example, consider a piston-like respose of a hillslope to an impulse of recharge (i.e. an instantaneous unit hydrograph, or IUH) given by:

$$IUH(t) = \frac{H(t) - H(t - L/V)}{L/V} \tag{1}$$

where $H()$ is the Heaviside step-function, L is the hillslope length, and V is the kinematic wave velocity. If the velocity V is randomized according to a log-normal distribution $f(V;\mu,\sigma)$ (as is typically found for the spatial variability of hydraulic conductivity) with parameters $\mu = \log V_0$ and σ, the resulting IUH becomes:

$$
\begin{aligned}
IUH(t) &= \int_0^\infty f(V;\mu,\sigma)\frac{H(t)-H(t-L/V)}{L/V}dV & (2)\\
&= \frac{V_0}{2L}e^{\frac{\sigma^2}{2}}\mathrm{Erfc}\left(\frac{\log\left(\frac{tV_0}{L}\right)+\sigma^2}{\sqrt{2}\sigma}\right) & (3)
\end{aligned}
$$

where Erfc() is the complementary error function. Figure 1 shows the form of this response for a variety of values of σ, representing different degrees of heterogeneity. This clearly shows that for small σ the piston-like response is preserved. For moderate σ the response becomes exponential, while for large heterogeneity, it becomes a power-law. Field studies of flow from hillslopes mostly report exponentially-tailed recessions [*Whipkey*, 1966, 1967; *Dunne*, 1978] suggesting perhaps that the degree of heterogeneity required to produce heavy tails is not common, or that such heterogeneous hillslopes are not selected for field studies.

Although illustrative, this simple mixing of distributions is inadequate to describe the behavior of heavy-tailed recessions, since it effectively assumes that the response is composed of an ensemble of pistons operating in parallel, each with a different characteristic velocity. In reality, an impulse of water moving through a heterogeneous hillslope will encounter a series of high and low conductivity areas which act as a series of time filters to the arrival of impulses at the base of a hillslope. Thus, the distribution of impulse arrivals is an integrated response that reflects the sum of waiting times applied to impulses traveling through a multitude of velocity zones. Heavy-tailed flow recessions imply heavily-tailed distributions of waiting times applied to individual impulses. The sum of independent and identically distributed (*iid*) heavy-tailed quantities (i.e., waiting times) converge in the limit to a Lévy-stable probability distribution [*Feller*, 1971; *Meerschaert* *and Scheffler*, 2001], which exhibit heavy tails. This rationale for heavy-tailed behavior has recently been used by *Ganti et al.* [2009a, this issue], *Ganti et al.* [2009b], *Hill et al.* [2009, this issue] and *Stark et al.* [2009, this issue] in the context of sediment transport.

This work describes the development of an analytical framework built upon this heavy-tailed time filtering that faithfully reproduces both the early- and late-time characteristics of heavy-tailed flow responses from heterogeneous hillslopes. Through the use of a mathematical technique termed *subordination*, the arrival of flow impulses at the base of a hillslope is randomized by replacing clock time with 'operational time', i.e. the time a flow impulse spends in motion. This process accounts for varying degrees of heterogeneity by incorporating, in the scaling limit, a heavy-tailed probability density of operational times with recharge impulses whose motion in linear (unsubordinated) time is governed by a kinematic wave equation.

A subordinated kinematic wave approach offers hope both as a closure relation for this flux, and as a template for the further development of closure relations of other fluxes. In this work we test this proposed method against numerical hillslope simulations of flow through random fields with various degrees of heterogeneity and several correlation length scales. This paper describes the theoretical development, parameterization and application of a subordinated kinematic wave equation to synthetic hillslope data. A quantitative link between the statistical properties of the heterogeneous random fields and the subordination framework is established through a systematic evaluation of flow responses given various degrees of heterogeneity.

2. Theoretical Development

To demonstrate the potential utility of the time subordination approach, we restrict our attention to the particular phenomenon of subsurface lateral flow over an impermeable base. In particular, we assume that the slope of this base is sufficiently steep that the topographic gradient is the dominant driver of flow. This allows us to ignore certain non-linearities in the flow (as discussed below), and base the theory on the kinematic wave equation proposed by *Beven* [1981], a simplified form of the modified Boussinesq equation. We will also make the assumption that recharge is uniform over the hillslope. In future work we hope to be able to relax some of these assumptions and broaden the framework.

The solution of the kinematic wave equation for a uniform, rectangular hillslope is a piston-like flow of impulse arrivals at the base of a hillslope. This condition simplifies application of the subordination approach, as a continuum of operational times can be assigned to individual impulses that would otherwise move at a constant velocity. Details of the kinematic wave equation, its applicability to steep hillslopes and subordination are provided below.

2.1. Kinematic Wave Equation

The Boussinesq description of lateral subsurface flow is based on a form of Darcy's law vertically integrated over the saturated zone, under the assumption that flow is primarily oriented parallel to an impervious hillslope base. Continuity in the saturated zone requires that:

$$\phi_d \frac{\partial h}{\partial t} = r - \nabla \left(\vec{q} \right) \tag{4}$$

where $h = h(x, y, t)$ is the thickness of the aquifer at a point given by co-ordinates (x, y) defined in the plane of the sloping base, at time t; ϕ_d is the drainable porosity and

$r = r(x, y, t)$ is the recharge rate. The components of the flux $\vec{q} = \vec{q}(x, y, t)$ in the i direction are given by:

$$q_i = -K_i h \left(\frac{dh}{dx_i} \cos\theta_i - \sin\theta_i \right) \tag{5}$$

which is a function of the hydraulic conductivity K_i, water table gradient dh/dx_i, and angle of depression θ_i in the i direction. Note that equation (4) is a modified form of the traditional Boussinesq equation which is purely diffusional and does not include a convective term.

The resulting equation is a non-linear PDE due to the multiplication of the h and dh/dx terms in the expression for the flux. However, conditions have been described under which this non-linearity can be ignored, most recently by *Harman and Sivapalan* [2009a]. The dh/dx term describes the component of the potential gradient driving flow given by the gradient of the aquifer thickness. When the ratio of the mean aquifer thickness $\bar{h}$ and the product of hillslope length L and slope $\tan\theta$ is small:

$$\bar{\eta} = \frac{\bar{h}}{L \tan\theta} \ll 1 \tag{6}$$

the potential gradient driving the flux is dominated by the topographic gradient. The expression for flux in the downslope direction in a straight, uniform hillslope (with i subscripts dropped, since flow is only downslope), then reduces to:

$$q = Kh \sin\theta \tag{7}$$

and the PDE is then linear. We can re-write this in terms of the velocity of an impulse moving through the hillslope v as:

$$q = vh\phi_d \tag{8}$$

where the velocity is defined by

$$v = \frac{K \sin\theta}{\phi_d}. \tag{9}$$

Considering only 1-D flow down the hillslope, the conservation equation for flow can then be written as:

$$\frac{\partial h}{\partial t} = -v\frac{\partial h}{\partial x} + \frac{r}{\phi_d}. \tag{10}$$

Beven [1981] used this approximation to derive an expression for the response of a sufficiently steep rectangular hillslope (with uniform conductivity) to an impulse of recharge applied uniformly across the hillslope. The discharge per unit depth of recharge is given by:

$$IUH_0(t) = \begin{cases} \dfrac{K \sin\theta}{L} & \text{for } 0 < t < \dfrac{L}{v} \\ 0 & \text{otherwise} \end{cases}. \tag{11}$$

Because the system is linear, this result is the impulse response function (or unit hydrograph), and the output of the system for an arbitrary timeseries of (spatially uniform) recharge inputs $r(t)$ can be obtained by convolving the unit hydrograph, equation (11), with the input:

$$\begin{aligned} Q_0(t) =& \int_{-\infty}^{\infty} r(\tau) IUH_0(t-\tau) d\tau \\ =& r(t) \star IUH_0(t) \end{aligned} \tag{12}$$

where the $\star$ represents the convolution operator. A build-up of storage in low-conductivity areas within heterogeneous hillslopes can create large gradients in the aquifer thickness, violating the assumption that these gradients have a negligible influence on flow. However, *Harman and Sivapalan* [2009b] demonstrated that these gradients do not have a significant effect on the flow when the condition in equation (6) is met. If we can assume that the behavior is still linear, it should be possible to find a new unit hydrograph for a heterogeneous conductivity field IUH_K that allows us to obtain the discharge result for arbitrary input in the same manner as for the uniform K field:

$$Q_K(t) = r(t) \star IUH_K(t). \tag{13}$$

We test this assumption using synethetic data generated from a numerical model that solves the full Boussinesq equation (equations 4, 5). The model is run first for an impulse of recharge to produce a 'unit hydrograph', and then for a hypothetical time-variable recharge event of 24 hours duration. By convolving the unit hydrograph with the recharge event, it should be possible to obtain a similar discharge timeseries as the numerical model if the hillslope is behaving linearly, and a different response if the linearity assumption is not valid.

2.2. Subordination of the Kinematic Wave Equation

The kinematic wave equation (equation 7) provides a basis for the development of an analytical framework for modeling flow through heterogeneous hillslopes. According to equation 7, the flux q [L/t] at the base of a hillslope for an instantaneous and evenly distributed recharge impulse of magnitude R for a hillslope of length L is a piston of magnitude Rv/L with duration L/v (Figure 2). The piston flow response arises from a collection of flow impulses at the hillslope base, where the impulses all move at the same velocity but have different starting locations. The arrival time of an individual impulse t_i with a travel distance of $L - x_i$, where x_i is the initial location and values of x increases in the direction of flow, is equal to $\frac{L-x_i}{v}$ (Figure 2). The study by *Harman* [2007] demonstrates that the motion of impulses within heterogeneous hillslopes is not subject to a constant or equivalent velocity. Rather, impulses are subject to a wide range of velocities as they migrate through a hillslope, and flow responses of heterogeneous hillslopes signify an assemblage of both fast and slow arrival times. As a consequence, the flow behavior of heterogeneous hillslopes clearly violates the equivalent velocity assumption of a piston. By randomizing the time the impulses spend in motion, and consequently arrival times t_i of impulses at the base of a hillslope via subordination, we broaden the applicability of equation 7 which was previously limited to only the least heterogeneous hillslopes.

Subordination is a standard tool in the theory of Markov and Lévy processes [e.g. *Baeumer and Meerschaert*, 2007; *Bertoin*, 1996; *Bochner*, 1949; *Feller*, 1971; *Meerschaert and Scheffler*, 2004, 2008; *Sato*, 1999]. The process of subordination refers to the replacement of linear time with operational time [e.g. *Baeumer and Meerschaert*, 2007; *Meerschaert and Scheffler*, 2004, 2008]. In our work, operational time refers to the random amount of time that a flow impulse 'operates' or participates in the motion process. Other earth science applications of subordination include transport of sediment particles in river systems [*Ganti et al.*, 2009b] and transport of solutes through heterogeneous aquifers subject to differential advection [*Baeumer et al.*, 2001] or retention in immobile zones [e.g. *Baeumer and Meerschaert*, 2007; *Benson et al.*, 2009; *Schumer et al.*, 2003].

We focus our application on hillslope flow responses with heavy-tails (i.e., exhibit a linear decay of flux over time on a log-log plot). The heavy-tails imply that the heterogeneity within a hillslope soil exerts a strong time influence on the motion of flow impulses; i.e. the time between motions is heavy tailed. However, it is well known that the classical advection-diffusion equation fails to take into account the heavy-tails in the waiting times or the velocities of transport [e.g. see *Benson et al.*, 2000, 2001, among many others]. It has been specifically shown that the governing equation for the transport with a heavy-tailed distribution with a power-law decay in the tails of waiting times and/or transport distances can be recast into a fractional advection-dispersion equation [e.g. *Meerschaert et al.*, 1999; *Meerschaert and Scheffler*, 2001; *Schumer et al.*, 2003]. Specifically, it has been shown that heavy-tails in waiting times can be taken into account by time fractional derivatives and heavy-tails in travel distances by space-fractional derivatives [e.g. *Benson et al.*, 2001; *Ganti et al.*, 2009a, this issue; *Schumer et al.*, 2003, 2009, this issue]. In the above case we focus our attention on heavy-tailed waiting times and in the scaling limit these motions (without recharge) satisfy the fractional-in-time differential equation:

$$\frac{\partial^\gamma h}{\partial t^\gamma} = -\frac{v\partial h}{s\partial x} \tag{14}$$

where the fractional time derivative of order γ and scale factor s (with units of [$\mathrm{t}^{\gamma-1}$]) describe an inverse stable distribution of waiting times between impulse motion. Note that the selection of a stable subordinator is motivated by mathematical limit theorems that describe the convergence of the sum of *iid* heavy-tailed random variables to a Lévy-stable density [*Meerschaert and Scheffler*, 2001]. In order to incorporate recharge $r(x,t)$ which modifies $\partial h/\partial t$, equation (14) needs to be recast (differentiating $(1-\gamma)$ times both sides in t) to expose a first order temporal derivative on the left hand side [*Baeumer et al.*, 2005]:

$$\frac{\partial}{\partial t}H(x,t) = -\frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}\frac{v\partial}{s\partial x}H(x,t) + \frac{r(x,t)}{\phi_d}. \tag{15}$$

Note that equation (15) is a generalized form of equation (10). Equations (10) and (15) are equivalent for the special case of $\gamma = 1$ and $s = 1$ which describes piston flow through a homogenous hillslope. The function H is given by:

$$H(x,t) = \frac{r}{\phi_d} \star \int_0^\infty h(x,u)p(u,t)\,du \tag{16}$$

where h is the piston flow solution to an impulse, and $p(u,t)$ is an inverse stable density of index gamma; i.e.

$$p(u,t) = \frac{t}{s^{1/\gamma}\gamma u^{1+1/\gamma}} g_\gamma\left(\frac{t}{(su)^{1/\gamma}}\right), \tag{17}$$

where g_γ is a standard stable density ($\int_0^\infty e^{-\lambda t} g_\gamma(t)\,dt = \exp(-\lambda^\gamma)$). Our approach essentially randomizes the response for a homogenous hillslope Q_0, equation (11), with a heavy-tailed subordinator to generate unit responses for heterogeneous hillslopes Q_K (equation (13)).

We can find the discharge (flux) F at a point x by looking at the rate of change of total mass to the right of x:

$$F(t,x) = \frac{\partial}{\partial t}\int_x^\infty H(\xi,t)\,d\xi = \frac{r}{\phi_d} \star \int_0^\infty \frac{1}{(su)^{1/\gamma}} g_\gamma\left(\frac{t}{(su)^{1/\gamma}}\right) \frac{\partial}{\partial u}\int_x^\infty h(\xi,u)\,d\xi\,du. \tag{18}$$

To model the flux of impulses at the base of a hillslope the following steps are required. First, an instantaneous recharge flux f_0 is distributed evenly for all x along a hillslope of length L according to:

$$f_0(x) = \frac{R}{L}\Phi(L-x)\Phi(x) \tag{19}$$

where R is total recharge applied to the entire hillslope, and $\Phi(\cdot)$ is the Heaviside step function. Then $r(x,t) = \delta(t)f_0(x)$. As $h(x,t) = \delta(x - vt)$, we obtain:

$$\begin{aligned}\frac{r}{\phi_d} \star \frac{\partial}{\partial u}\int_x^\infty h(\xi,u)\,d\xi =& r \star \frac{\partial}{\partial u}\Phi(vu - x) = r \star v\delta(vu - x)\\ =& v f_0(vu).\end{aligned} \tag{20}$$

Hence equation (18) becomes:

$$\begin{aligned} F(t, L) &= \int_0^{(L/v)} \frac{Rv}{L\phi_d} g_\gamma \left(\frac{t}{(su)^{1/\gamma}} \right) (su)^{-1/\gamma} \, du \\ &= \int_0^{s(L/v)} \frac{Rv/s}{L\phi_d} g_\gamma \left(\frac{t}{u^{1/\gamma}} \right) u^{-1/\gamma} \, du \end{aligned} \tag{21}$$

where an $x = L$ substitution describes the flux of recharge impulses at the base of the hillslope and v/s is an effective velocity used in calibration to the numerical data. Dividing this by the size of the original impulse gives the dimensionless unit hydrograph (equation 13):

$$IUH_K(t) = \frac{F(t, L)}{R} = \int_0^{s(L/v)} \frac{v/s}{L\phi_d} g_\gamma \left(\frac{t}{u^{1/\gamma}} \right) u^{-1/\gamma} \, du. \tag{22}$$

Parameters γ and s will depend on the properties of the hillslope, though analytical expressions for this relationship have not been derived at this time. Solutions to the subordinated kinematic wave equation (equation 21) for a wide range of γ are shown in Figure 3. Values of γ describe the tail index of a $\gamma-$stable subordinator used to regulate the times that the piston impulses participate in the motion process. As values of γ decrease, more probability mass is shifted to the tail of the subordinator and the likelihood of extreme events increases. This translates into shorter amounts of time that the recharge impulses actively spend in motion, greater impulse arrival times at the hillslope base, and greater deviations from the homogeneous piston flow response (Figure 3). If heterogeneity imparts a temporal memory on the motion of recharge impulses in hillslopes, then increases in the heterogeneity should be linked to decreases in γ.

The mass of the initial piston impulse is conserved by equation (21) for all parameter values. The subordinated solution at early time has a decay rate of $-1 + \gamma$. Once the piston impulse is depleted, the decay of the subordinated solutions is more rapid (i.e., steeper slope) and transitions to $-1 - \gamma$. Transition times between the two slopes depend on values of both γ and s. For example, the transition time for $\gamma = 0.9$ when $s = 1$ is close to the duration of the classical piston (γ=1) (Figure 3). Lower values of γ temporally spread the mass of the piston, and slope transitions are not observed in Figure 3 when $\gamma \leq 0.3$.

Calibration of the analytical solutions to data is facilitated by changing values of s to apply a temporal scale to the velocity of the piston input. This temporal scale changes the velocity of the piston, thereby changing the magnitude and duration of the piston input (Figure 3). The quotient of v and s can be viewed as an overall or 'effective' velocity, where $s > 1$ and $s < 1$ represent lower and greater velocities than originally assigned to the piston, respectively (Figure 3). If $s = 1$, the original piston velocity and effective velocity are equal. This method of varying s allows us to efficiently calibrate the subordinated solution to the numerical data without changing the piston parameters directly. Since the hydraulic conductivity fields are random and all other hillslope parameters are held equal, the use of s is equivalent to dividing the mean piston K value (and hence v) by s.

3. Numerical Simulations of Hillslope Flow

Numerical simulations of hillslope flow are used to provide synthetic data for three purposes: (1) to verify that the linearity of the kinematic wave condition that is known to hold when the conductivity field is uniform also holds for variable conductivity fields; (2) to determine whether the subordination approach can be used to simulate the effect of heterogeneous conductivity fields on the impulse-response of a hillslope; and (3) to determine whether the fitted impulse response can be used as an instantaneous unit hydrograph (IUH) to reproduce the discharge from a hillslope subject to a realistic recharge event.

The response of hillslopes to impulses and time-variable recharge events were generated using the model described in *Harman and Sivapalan* [2009a] and *Harman and Sivapalan* [2009b]. This model solves the full Boussinesq equation (4 and 5) on a 64×64 square grid using a finite volume scheme with adaptive timestepping. The simulated hillslope used for most of the simulations measured 100m×100m and had a constant slope of 0.1m/m, with a zero flux boundary condition at the upper boundary, periodic boundary conditions on the left and right (i.e. flux out of one side enters the other side, and gradients are calculated as though indentical hillslopes lay side-by-side), and a kinematic boundary condition at the base (i.e., constant potential gradient). The soil drainable porosity was set at 0.34 and the mean log conductivity $\langle \log K \rangle$ at 0.1m/hr. These physically realistic values were chosen for convenience, rather than to represent any particular hillslope.

The natural log of the field of conductivity values was assumed to have a Gaussian distribution. Four types of correlation structure for the log conductivity were used (Figure 4). Three were random exponential fields with correlation lengths (normalized by the domain size) of 1/20, 1/10, and 1/5 representing short, intermediate, and long range correlations, respectively. A fourth ('combined') case with multiple correlation lengths was constructed by averaging the values of $\log K$ from three independently generated correlation fields with correlation lengths of 1/20, 1/10, and 1/5. The incorporation of multiple correlation lengths into a single field was motivated by studies of subsurface heterogeneity that report spatial variations in K on a multitude of scales [e.g. *Neuman and Di Federico*, 2003, and references therein]. Though studies of subsurface heterogeneity within hillslope soils are limited, *Kirchner et al.* [2001] reported a characteristic dispersion length scale for the Plynlimon catchment that approached the entire length of the hillslope.

This finding suggests, since the dispersion length scale is directly influenced by the length scale of variability of the hydraulic conductivity field, that the Plynlimon hillslope soils exhibit spatial variability in K at all scales up to the length of the entire hillslope. Note that the averaging of the three fields was performed on the Gaussian random variables before exponentiating. Three degrees of variability in $\log K$ were used initially: $\sigma_{\log K} =$ 1, 5, and 10. A fourth value of $\sigma_{\log K} = 3$ was added later exclusively for the combined correlation length case.

For the impulse responses, 100 realizations of each type of heterogeneity field were simulated. The hillslopes were initialized with a uniform 34mm of drainable water (or a saturated thickness of 100mm) at time zero and allowed to drain with no further input. This corresponds to a dimensionless thickness of $\bar{\eta} = 0.01$, making the flow strongly controlled by the topographic gradient, and in the kinematic range of dynamics [*Harman and Sivapalan*, 2009a]. Simulated discharge was calculated at logarithmically spaced points in time ranging from 1 second to several years (though due to computational constraints in the $\sigma_{\log K} = 10$ case not all 100 runs completed the full duration). The ensemble geometric mean and standard deviation was calculated at each point in time.

The discharge response to a time-variable recharge event was calculated using 100 random field realizations with $\sigma_{\log K} = 5$ and a multiple correlation length structure. The storage in this case at time $t = 0$ was set to zero, and recharge was then applied uniformly over the whole grid from time zero. The same hourly time series of recharge was used in each simulation, generated initially by multiplying a Gaussian bell-curve by a set of 24 uniformly distributed random numbers. Recharge over each 1 hour period was assumed to be constant. Output, consisting of discharge at the base of the hillslope and storage within the random field, was generated and recorded at 1 minute time steps.

4. Results

The numerical simulations were used to generate flow responses for heterogenous hillslopes subject to single and multiple impulse events. The simulated hillslope responses serve as synthetic data to test the assumption of linearity in hillslope response, and to ascertain the suitability of, and any potential limitations pertaining to, the use of a subordinated kinematic wave equation for heterogeneous hillslopes. The simulation of flow through hillslopes with a wide range of heterogeneity and correlation structure allows us to systematically investigate flow responses and parameterize equation (21).

4.1. Linearity of Hillslope Response

The impulse responses for each realization of the conductivity field were numerically convolved with the time-variable recharge event at a 1 minute time step, and the result compared with the numerically modeled response to the recharge event. The results for six typical realizations of the conductivity fields with $\sigma_{\log K} = 5$ and the combined correlation lengths are shown in Figure 5. In every case there is excellent correspondence between the hydrograph predicted by the full non-linear Boussinesq model and that predicted by the convolution of the input with the impulse response. This correspondence would not be possible if non-linear interactions were significant. We can therefore be confident that the assumption of linearity used in the subordinated piston approach is appropriate for hillslopes with $\bar{\eta} \ll 1$.

4.2. Single Impulse Response

The single impulse data describe the hillslope response to a single, instantaneous impulse of 34mm applied at $t = 0$, and provide the foundation for our assessment and parameterization of a subordinated kinematic wave equation. Discharge at the hillslope base for all 100 realizations for the initial σ_{LogK} and correlation structures are shown shown in Figure 6. We display our results on a logarithmic scale because of the orders of magnitude in simulation time and our interest in heavy-tails.

The mean impulse response is insensitive to the correlation length regardless of σ_{LogK} values. This is clearly evident when the ensemble mean of all cases are plotted together (Figure 7). However, the variability around the mean is sensitive to correlation structure, particularly for early time (Figure 6). Discharge at early time varies over several orders of magnitude for $\sigma_{\log K} = 1$ for the large correlation length case ($L/5$ or 20m), but is mostly constrained to a single order of magnitude for small correlation length ($L/20$ or 5m). The combined correlation length case has a similar range of values as the large correlation length, but variability in individual realizations (as shown by the ± 1 standard deviation) is generally more tightly constrained. The distribution of log discharge has a negative skew, with more small values far from the mean than large values. Particularly low values tend to have a constant discharge for early times.

The subordinated solutions were fitted to the ensemble mean discharge of the multiple correlation structure. Mean discharge curves for all correlation structures are more or less equal (Figure 8) indicating that the selection of one correlation structure over the others has negligible impact on the results of our study. All best-fit parameters are based on visual inspection with the objective of matching both early- and late-time portions of the flow response (Table 1). The piston impulse subject to subordination is defined according to quantifiable properties of the synthetic hillslopes and the impulse including mean K, drainable porosity, slope, and impulse magnitude. These values are held constant for all cases.

Mild heterogeneity is represented by the K fields with $\sigma_{LogK} = 1$ (Figure 8, top left). Impulses traveling through these random fields produce near piston flow responses as expected from kinematic wave theory with a slight exception. Rather than an instantaneous drop occurring exactly at L/v and a flux magnitude of Rv/L, the hillslope response exhibits a slightly lower, near-piston flux along with an exponential tail. Since the numerical simulations conserve mass, the slight decrease in flux from the theoretical piston is caused by the exponential tail. The absence of a power-law tail in the response produced by these fields excludes the use of a subordinated kinematic wave equation. However, a best-match kinematic hillslope equation is included for comparison (Table 1). As expected when comparing to an incompatible model (a subordination model with power-law tails versus simulated data with exponential tails), the fit between the subordinated solution and the mild heterogeneity case is poor.

Moderately to highly heterogeneous hillslopes with $\sigma_{LogK} \geq 3$ (Figure 8) exhibit heavy-tailed hillslope responses. For these cases, the match between the numerical data and subordinated solutions are excellent. Not only are the early and late time behavior well-represented, the trend of the numerical data supports the subordinated kinematic wave theory that predicts a slope transition from early time decay of $-1+\gamma$ to $-1-\gamma$ at later times once the piston impulse is depleted. The transition between the early and late time regimes represents some 'characteristic' time scale over which the subordinated solution spreads the mass of the piston. If this slope transition did not occur, mass would not be conserved in the subordinated solution. In addition to the excellent fit between the numerical data and subordinated solution, values of γ decrease with increasing heterogeneity (σ_{LogK}). As values of σ_{LogK} increase, decreases in γ are accompanied by decreases in s (Table 1). The implications of these observations are discussed later.

4.3. Event Response

The subordinated kinematic wave equation proved successful for the single impulse case. The next logical step in our analysis is to evaluate subordinated solutions against an event with time-variable recharge. This is more realistic than an instantaneous impulse as storms vary in intensity and have a temporal duration. The flow response to a time-variable recharge event with a total input depth of 20mm distributed over 24 hours with a peak of 3.2mm/hr was simulated for 100 realization of spatial variability in the conductivity field. The case of $\sigma_{LogK} = 5$ was chosen due to computational speed relative to the $\sigma_{LogK} = 10$.

The subordinated solution was generated by subordinating a piston with the subordination parameters, γ and s, obtained from the single impulse, $\sigma_{LogK} = 5$ case (Table 1). The subordinated solution provides an excellent prediction of the geometric mean discharge from the hillslope for a recharge event (Figure 9). There is a slight under-prediction of the peak, but the recession behavior is very well captured. The peak modeled discharge from the ensemble of hillslopes varies from 0.3mm/hr to 0.002mm/hr, with a geometric mean of 0.096mm/hr. The subordinated solution has a mean peak discharge of 0.080mm/hr. However from the end of the recharge event to the end of the modeled period (5 days) the root-mean-square error (RMSE) is less than 3.2x10^{-4}mm/hr. Note that the subordinated solution used for the recharge event input is based on subordination parameters obtained from the impulse flow response with $\sigma_{LogK} = 5$, and not by fitting them to this hydrograph.

5. Discussion

The kinematic wave equation was successfully extended to the case of heterogeneous hillslopes through the incorporation of a stable subordinator. Solutions to this equation were found to provide excellent early and late time matches to numerical data for moderately to highly heterogeneous hillslopes ($\sigma_{LogK} \geq 3$) for both single impulse and realistic time-variable recharge event scenarios. Measurable hillslope and impulse parameters in all cases were used to define the piston, and subordinated parameters γ and s were based on matching to early and late time characteristics of the numerical flow data.

Results from this study have several significant implications. First, the results suggest that a definitive correspondence exists between γ and hillslope heterogeneity. By using the standard deviation of hydraulic conductivity of the random fields as a quantitative measure of heterogeneity, the following proposed relationship:

$$\gamma = 1.00e^{-0.165\sigma_{LogK}} \tag{23}$$

suggests a strong ($R^2 = 0.99$) exponential correlation between these parameters (Figure 10). We acknowledge that this fit is only based on four data points which may overestimate the correlation. However, this relation albeit preliminary is very exciting and provides a link between heterogeneity and model parameters in a stochastic flow equation. Such links have proved elusive in fractional-derivative extensions to groundwater equations. Only a handful of theoretical contaminant transport studies have linked physical properties of porous and fractured media to the order of spatial [*Benson et al.*, 2000;

Reeves et al., 2008a, b] or temporal [*Zhang et al.*, 2007] derivatives in fractional advection-dispersion equations (fADEs). Derivations of fADEs have some similarity with this work as fractional-order derivatives invoke Lévy processes described by γ-stable probability densities.

Second, the predominance of heavy tails in the heterogeneous fields shows that increases in heterogeneity, while influencing early impulse arrivals and piston velocity, predominately increase time delays between impulse motion (i.e., retention). This is demonstrated by the heavy tailed decay of impulse flux simulated at the base of the hillslopes (Figure 8). Thus, heterogeneity exerts a strong influence on the characteristic time scale required to dissipate the mass of the piston.

Third, the numerical results suggest that correlation structure does not play a major role in the ensemble flow response, though correlation structure was found to exert minor differences in flow about the mean. An explanation for the insensitivity of correlation structure on flow response is unknown at this time. A suggested rationale is that flow impulses are predominately delayed in heterogeneous hillslopes, and these delays are observed through increases in storage within low velocity zones. If the relative proportion of low velocity zones is equal for random fields with the same value of σ_{LogK}, these fields may have similar storage characteristics regardless of correlation structure.

Finally, it should be noted that although these results are promising, their application is limited to cases where the underlying assumptions are valid. In particular, the assumption of kinematic flow will be valid only in particularly steep, conductive hillslopes. Field studies such as *Dunne* [1978], have long shown that in areas with lower gradients the mounding of the subsurface saturated layer at the hillslope base can create pressure gradients that are a significant driver of lateral flow (see also the model of *Cayar and Kavvas* [2008]). We have also assumed that the soil column is sufficiently deep to accommodate the saturated layer, and so do not make any predictions concerning the location of surface exfiltration [*Dunne*, 1975, 1978] which can be a significant control on surface saturation [*Ragan*, 1968; *Dunne*, 1975]. The assumption of a planar, uniform slope for the base does not reflect the convergent and divergent topographies often found in natural hillslopes, which play an important role in the focusing of flow and the production of surface saturation [*Dunne*, 1978; *Kavvas et al.*, 2004; *Chen et al.*, 2004]. Recharge to the hillslope aquifer may not be uniform as we have assumed, and may be correlated in some way to the heterogeneity of aquifer properties, since it also must percolate through the soil medium [e.g. *Maxwell and Kollet*, 2008]. Other models including the WEHY model (e.g., *Kavvas et al.*, 2004) allow for spatially-variable recharge.

6. Conclusions

A subordinated kinematic wave solution, where the time that flow impulses spend in motion is randomized, faithfully reproduces both the early and late time characteristics of heavy-tailed hillslope flow responses generated by numerical simulations of heterogeneous hillslopes when $\sigma_{LogK} > 1$. Thus, it provides a parameterization of the flow at the scale of the entire hillslope that accounts for the effects of heterogeneity without resolving them explicitly. This is the condition that must be met by the flux closure relations required for Representative Elementary Watershed (REW) -type models [*Reggiani et al.*, 1999, 2000]. Ideally, closure relations for hydrologic fluxes will consist of parameters that are physically meaningful and measurable at the scale of the hydrologic element they are designed for. In this case we have shown that the effects of heterogeneity can be accounted for in a hillslope scale closure relationship with the introduction of a stable subordinator that requires only two additional parameters, γ and s. Our results suggest that the subordination parameters are linked to physical properties of the hillslope including mean and log standard deviation of hydraulic conductivity, drainable porosity, hillslope angle and length, and magnitude of recharge impulse. Only a single time shift parameter, representing effective velocity, is needed to calibrate or scale the numerical data to the subordinated solution. Thus, it is possible that the subordination approach offers a closure relationship for kinematic subsurface flow through heterogeneous hillslopes provided that the degree of heterogeneity present in the conductivity field is measurable. Further work is required to test the model against field data from well-studied hillslopes.

This work offers an exciting new avenue for research, particularly as it builds on existing theory developed for the case of uniform hillslopes. Recent work by *Troch et al.* [2004] and *Troch et al.* [2002] (and others) has developed analytical expressions for the unit hydrograph of subsurface flow through hillslopes with a wide range of geometries (including convergent and divergent planforms) and flow dynamics from kinematic wave to fully diffusive. An obvious extension of this work is therefore to examine whether it can be used in such circumstances. If so it could pave the way for a general theory of subsurface flow closure in heterogeneous hillslopes.

The subordination approach may also find application in modeling the vadose zone flow in certain cases. Kinematic wave theory has also been applied to model the infiltration of water into macropores [*Beven*, 2006], and is the basis of the MACRO model of preferential infiltration [*Larsbo et al.*, 2005]. These models generally assume a single 'effective' celerity of flow. Perhaps a better match to observed data could be found by using a subordination approach to account for the variability of flow velocities through macropores. This is left for future work.

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Table 1. Fitted parameters of the subordination kinematic wave equation to mean impulses responses from ensemble hillslope simulations with various degrees of heterogeneity.

$\sigma_{\log K}{}^1$	γ^2	s^3	$\langle V\rangle^4$
1	0.9	1.00	0.12
3	0.6	0.30	0.4
5	0.4	0.07	1.7
10	0.2	0.02	6.0

[1] Natural log of the standard deviation of K in the random hillslope fields.

[2] Tail index of stable subordinator.

[3] Temporal shift applied to piston during calibration.

[4] Effective velocity $[L/t^{\gamma}]$ of subordinated solution: ratio of v and s, where $v = 0.12$ [m/hr] for all single impulse realizations.

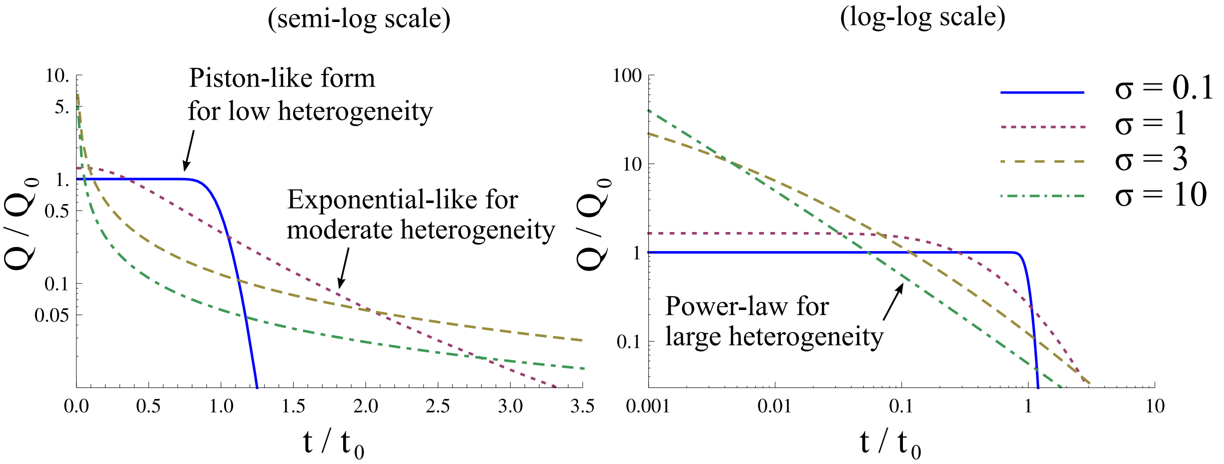

Figure 1. Impulse response obtained by randomizing the velocity of a piston according to a log-normal distribution with various σ parameters. This approximates the subsurface discharge response of a hillslope comprised of strips (from ridge to base) of diffferent soil materials. Though unrealistic, it demonstrates how the same underlying kinematic-wave like behaviour can produce piston-like, exponential-like and power-law-like behavior, depending on the degree of spatial variability.

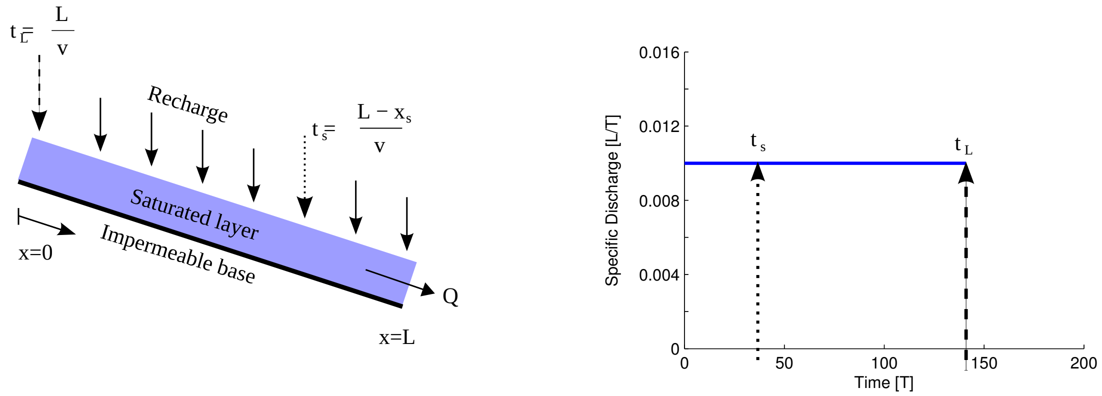

Figure 2. Schematic depicting how recharge impulses distributed along a hillslope (left) with constant velocity produce a piston flow response at the base of a hillslope (right), where the magnitude of the piston is Rv/L with duration L/v. An impulse at an arbitrarily initial location x_s has an arrival time t_s of $(L - x_s)/v$.

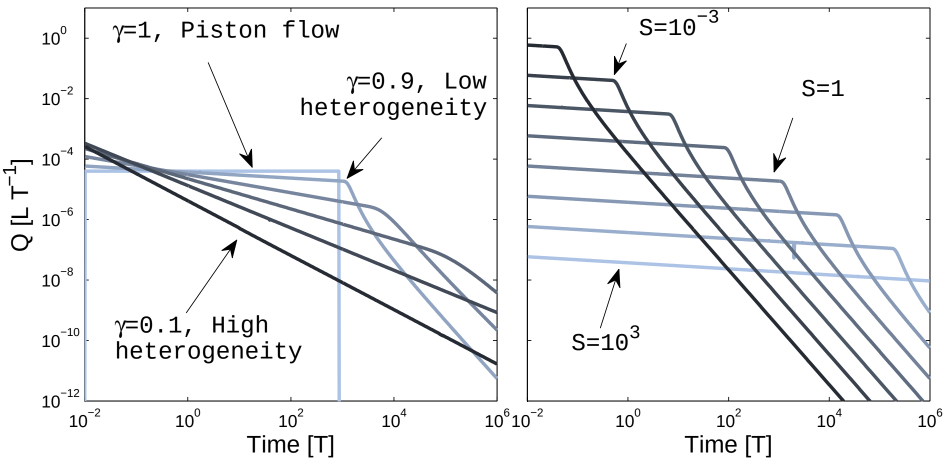

Figure 3. Kinematic wave solutions for (left) values of γ ranging from 0.1 (highly heterogeneous) to 1 (homogenous, classic piston) with s held constant at 1 and (right) for values of s ranging from 10^{-4} to 10^{3} with γ held constant at 0.9. The subordinated solution at early time has a slope equal to $-1+\gamma$ under the influence of the piston impulse and transitions to a slope of $-1-\gamma$ after depletion of the piston impulse input. Times of transition between the two slopes are directly related to s and inversely related to γ.

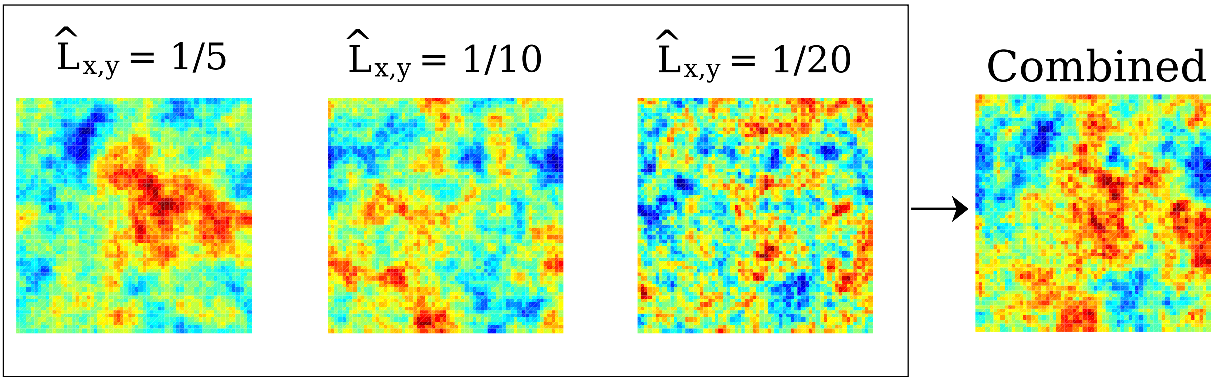

Figure 4. Examples of random K fields with $\sigma_{LogK} = 5$ given correlation length scales (relative to the domain size) of 1/20, 1/10 and 1/5. The 'Combined' case was computed by averaging the three previous fields, thereby containing multiple correlation lengths.

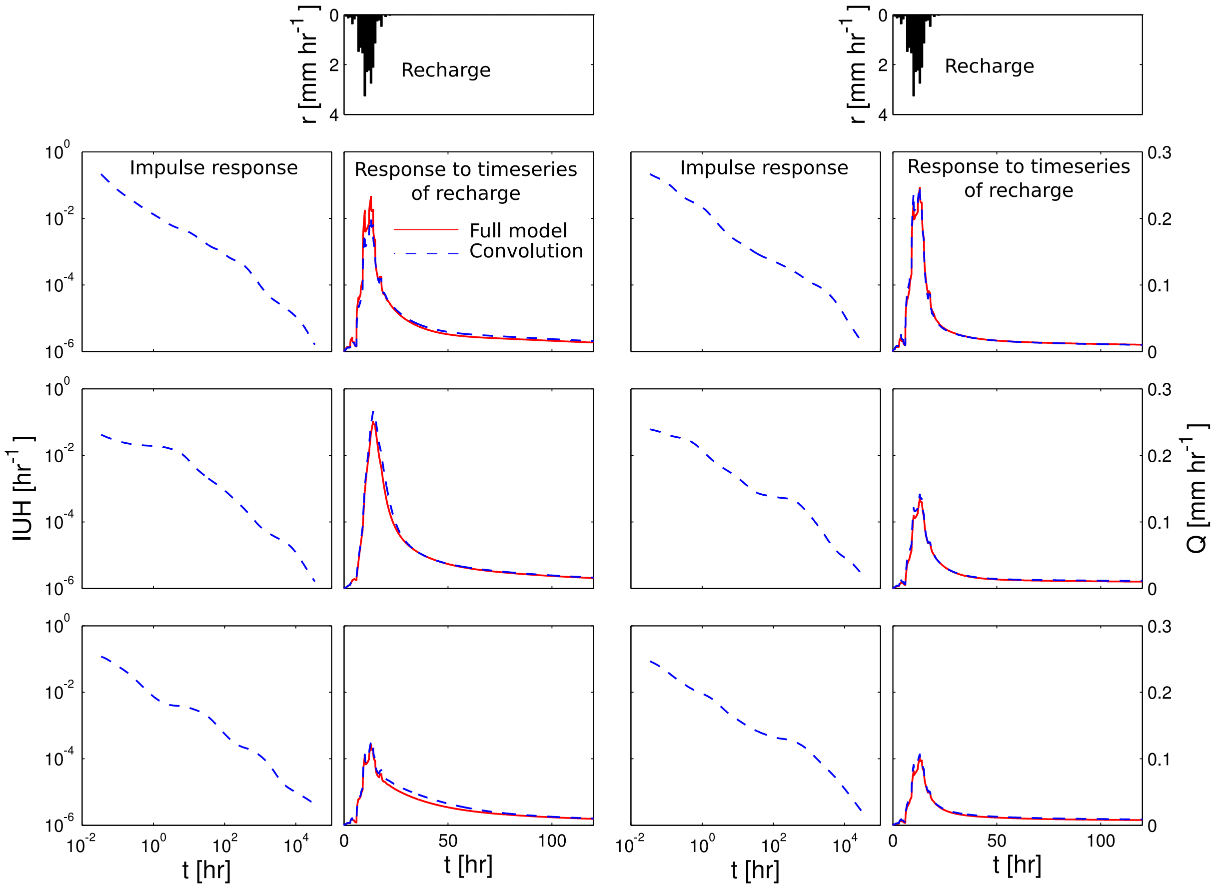

Figure 5. Evidence for the approximate linearity of the subsurface flow response described by the Boussinesq equation in highly heterogeneous conductivity fields. The left plot of each pair shows the impulse response, and the right shows the response to a arbitrary input predicted by the full Boussinesq model (solid line) and predicted by convolving the impulse response with the (dashed line) recharge input. Details given in the text.

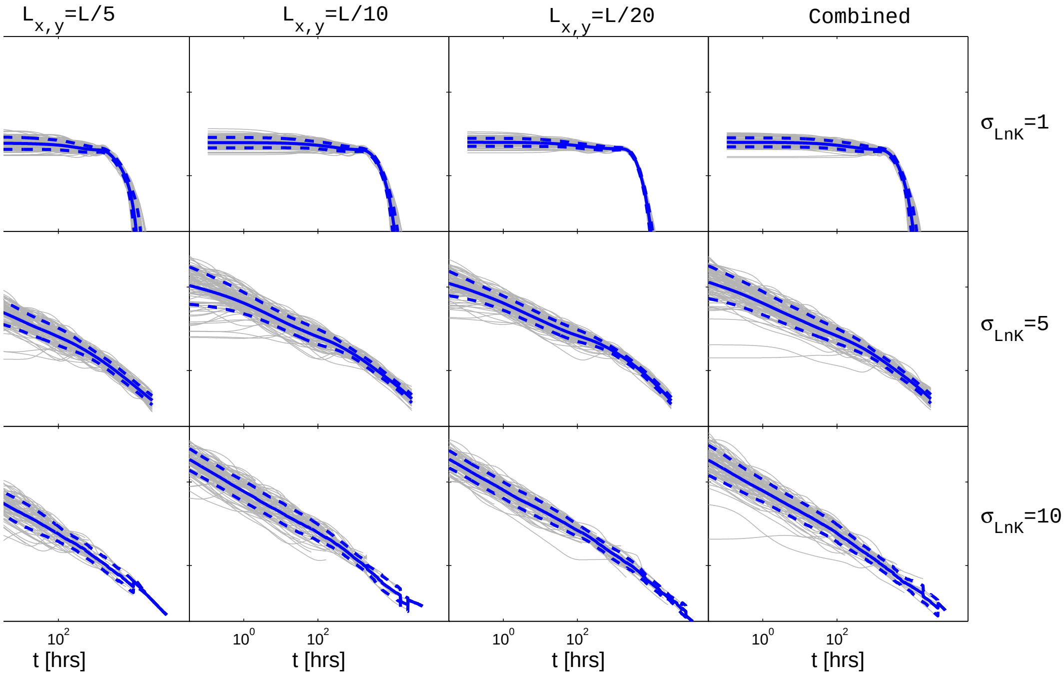

Figure 6. Discharge results with time for (top to bottom) $\sigma_{LogK} = 1$, 5, and 10 and (left to right) $\hat{L}_K = 1/5, 1/10, 1/20$ and the combined case. Solid line is the log mean, and dashed lines are ± 1 standard deviation of the log transformed values.

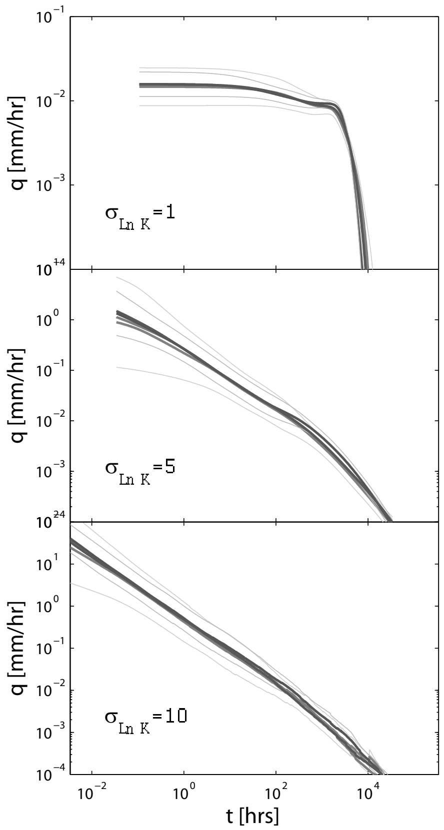

Figure 7. Mean discharge for (top to bottom) $\sigma_{\ln K} = 1$, 5, and 10 for each of the correlation length cases. The correlation length seems to have very little effect on the ensemble (geometric) mean discharge, but does affect the standard deviation of discharge; $\pm 1\sigma$ is plotted in light gray for correlation lengths $L/5$ and $L/20$.

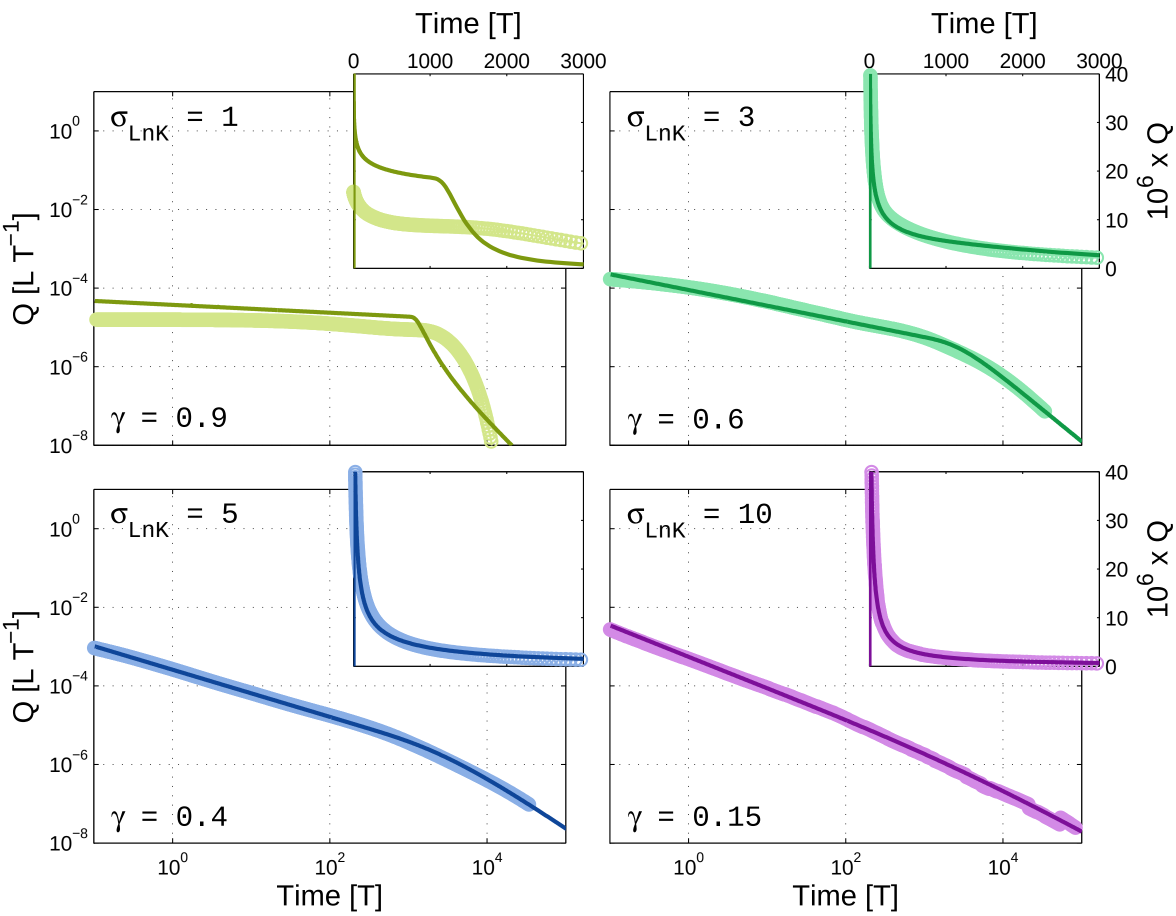

Figure 8. Ensemble mean discharge for σ_{LogK} values of 1, 3, 5 and 10, respectively, along with best fit subordinated wave equation, plotted on log-log axes. Note that the absence of heavy tails in the $\sigma_{LogK} = 1$ case excludes the use of the kinematic wave equation. However, its solution is provided for comparison. Insets show same data in with linear axes.

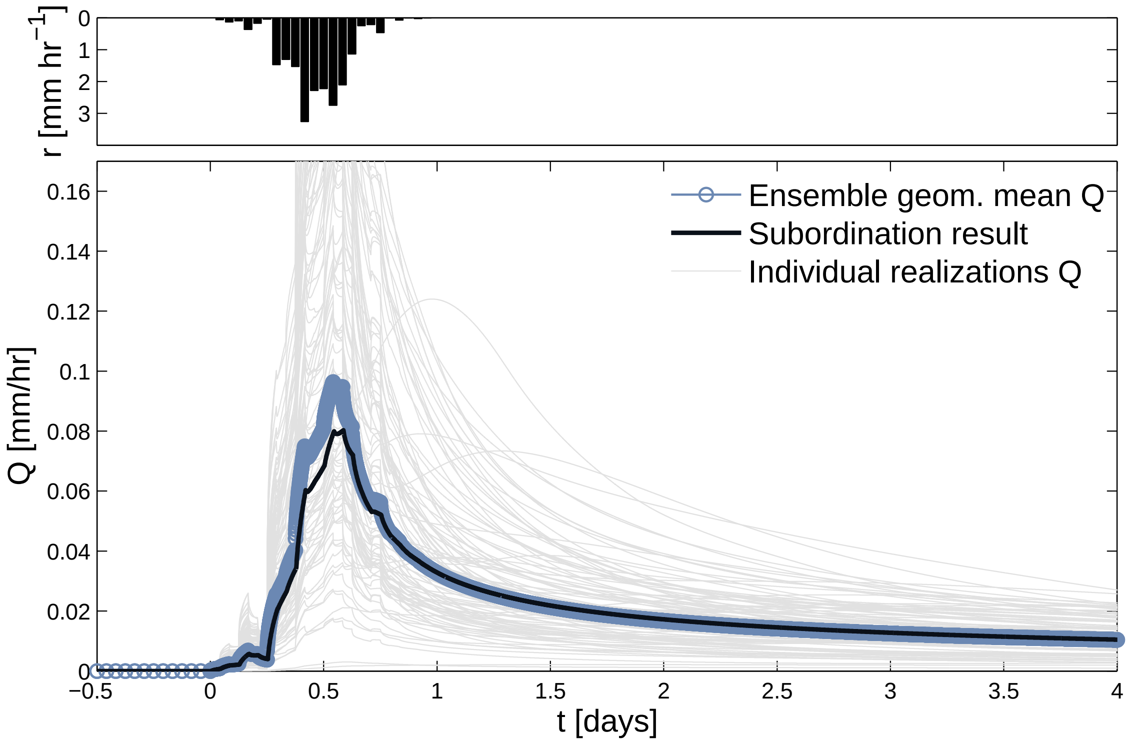

Figure 9. Simulated discharge from 100 generated heterogeneous hillslopes ($\sigma_{\log K} = 5$, 'combined' correlation lengths) subject to a 24 hour recharge event, and the prediction provided by the subordination approach (black line). Gray lines give individual realizations of Q, dark gray is their geometric mean. The recharge event is shown at the top.

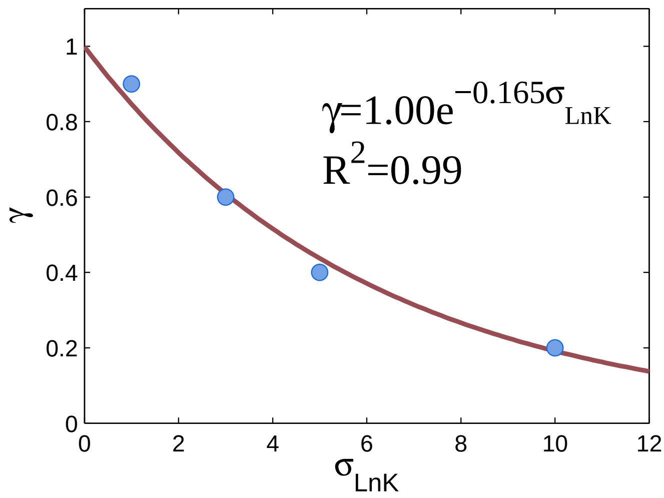

Figure 10. Empirical relationship between heterogeneity and tail index of the γ-stable subordinator.