A higher order resolvent-positive finite difference approximation for fractional derivatives on bounded domains

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Abstract

We develop a finite difference approximation of order α for the α-fractional derivative. The weights of the approximation scheme have the same rate-matrix type properties as the popular Grünwald scheme. In particular, approximate solutions to fractional diffusion equations preserve positivity. Furthermore, for the approximation of the solution to the skewed fractional heat equation on a bounded domain the new approximation scheme keeps its order α whereas the order of the Grünwald scheme reduces to order $\alpha - 1$, contradicting the convergence rate results by Meerschaert and Tadjeran.

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1 Introduction

In [1, 2] it was shown that for any $1 < \alpha \le 2$, the Cauchy problem

$$u'(t,x) = \begin{cases} \left(\frac{d}{dx}\right)^\alpha u(t,x) & x \in (0,1) \\ 0 & \text{else} \end{cases}, \tag{1.1}$$

with initial condition $u(0,x) = u_0(x)$ is well-posed on $L^1(0,1)$ and $C_0(0,1)$ and the solution was approximated with a backward Euler (time)/Grünwald (space) scheme. It was observed that as soon as the solution has mass near the left boundary the apparent order of the spatial approximation scheme reduced to order $\alpha - 1$ due to the fact that at that stage the eigenfunction

$$u_c(x) = \sum_{n=1}^{\infty} \frac{c^{n-1} x^{n\alpha-1}}{\Gamma(n\alpha)}$$

(with $c < 0$ largest such that $\sum_{n=1}^{\infty} \frac{c^{n-1}}{\Gamma(n\alpha)} = 0$) becomes dominant and behaves like $x^{\alpha-1}$ near the left boundary. The fact that there are α-times differentiable functions in $C_0(0,1)$ whose zero extension to the real line is not α-times differentiable was overlooked in [3, 4] while arguing for first-order convergence of the Grünwald scheme. In order to deal with this boundary behaviour we developed a new spatial scheme

$$A_h^\alpha f(x) = \frac{1}{h^\alpha} \sum_{n=-1}^{\infty} w_{n+1} f(x - nh) \tag{1.2}$$

that turns out to be of order α for all 2α-times differentiable functions as well as for the spatially semidiscrete approximation of (1.1).

The paper is organized as follows.

In Section 2 we introduce the scheme, state the main results, and prove basic properties of the weights w_n.

In Section 3 we first consider the problem on $\mathbb{R}$ and show that the scheme $A_h^\alpha f \to \left(\frac{d}{dx}\right)^\alpha f$ converges with order $\beta \le \alpha$ on all spaces where the shift group is strongly continuous and f is $\alpha+\beta$-times differentiable. We furthermore show that A_h^α generate uniformly analytic semigroups.

In Section 4 we prove that the scheme provides convergence of order α to solutions to (1.1). We use a piece-wise power interpolation, which is exact for $x^{\alpha-1}$, to obtain this rate of convergence for all x, not just on grid points.

2 The scheme

In order to avoid convergence-destroying residuals from approximating the fractional derivative of $x^{\alpha-1}/\Gamma(\alpha)$ (the δ-function) we are looking for weights

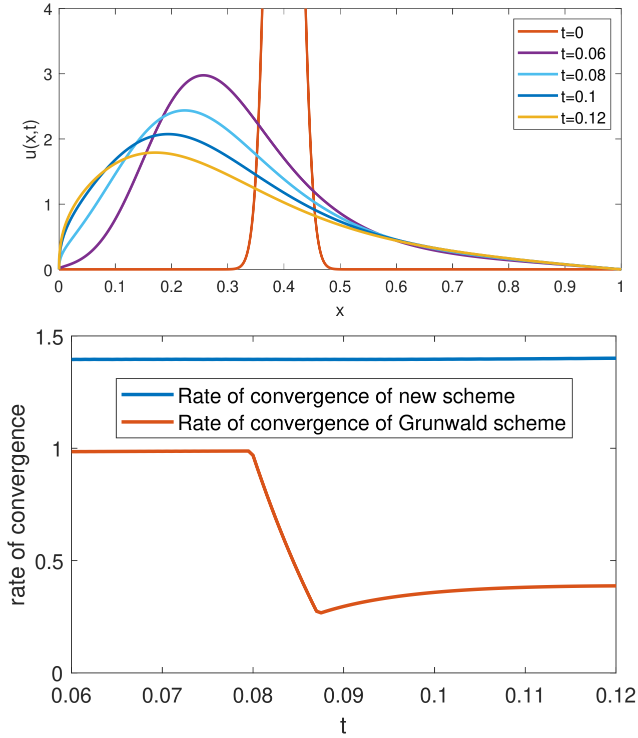

Fig. 1 Plot of approximate solutions and observed error rate to (1.1) for $\alpha = 1.4$ and $u_0(x) = \exp(-(x-0.4)^2/2\sigma^2)/\sqrt{2\pi\sigma^2}$ with $\sigma^2 = 0.0005$. The relative error for $n = 400$ is about 0.3% for the new scheme and 2% for the Grunwald scheme.

that make the scheme (1.2) *exact* for $x^{\alpha-1}$; i.e. we are looking for weights such that (1.2) gives exactly $1/h$ at node 0 if applied to $x^{\alpha-1}/\Gamma(\alpha)$. In particular,

$$\frac{1}{h^\alpha}\begin{pmatrix} w_1 & w_0 & 0 & \dots & 0 \\ w_2 & w_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & w_1 & w_0 \\ w_n & \cdots & \cdots & w_2 & w_1 \end{pmatrix} \begin{pmatrix} 0 \\ h^{\alpha-1} \\ (2h)^{\alpha-1} \\ \vdots \\ \vdots \\ ((n-1)h)^{\alpha-1} \end{pmatrix} = \begin{pmatrix} \Gamma(\alpha)/h \\ 0 \\ \vdots \\ \\ \vdots \\ 0 \end{pmatrix}. \tag{2.3}$$

Multiplying both sides by h and shifting the vector one up (deleting the first column of the matrix) we obtain the easily and quickly solvable tri-diagonal system determining the weights w_j,

$$\begin{pmatrix} w_0 & 0 & \dots & 0 \\ w_1 & w_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ w_{n-1} & \cdots & w_1 & w_0 \end{pmatrix} \begin{pmatrix} 1 \\ 2^{\alpha-1} \\ \vdots \\ n^{\alpha-1} \end{pmatrix} = \begin{pmatrix} \Gamma(\alpha) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.4}$$

Our two main results regarding the scheme (1.2) are as follows.

Theorem 2.1 *Let $X = C_0(\mathbb{R})$ or $X = L^p(\mathbb{R})$, $1 \le p < \infty$. Let $A = d/dx$ and $1 < \alpha \le 2$ and $0 < \beta \le \alpha$. Then there exists $C > 0$ such that for $f \in D(A^{\alpha+\beta})$,*

$$\|A^\alpha f - A_h^\alpha f\| \le Ch^\beta \|A^{\alpha+\beta} f\|,$$

where $A_h^\alpha f$ is the scheme defined in (1.2) using the weights satisfying (2.4).

Proof See Section 3. □

We denote with $C_0(\Omega)$ the completion with respect to the supremum norm of the space of continuous functions that are compactly supported within Ω; i.e. $C(0,1)$ is the space of continuous functions f that satisfy $f(0) = f(1) = 0$ and $C(0,1]$ is the space of continuous functions f satisfying $f(0) = 0$.

On a bounded domain we have the following result regarding the solution semigroup $\{S(t)\}_{t\ge0}$ to (1.1) on $C_0(0,1)$ and the matrix semigroup $\{\mathbf{S}_h(t)\}_{t\ge0}$ solving its finite difference approximation

$$\frac{d}{dt}\vec{u}_h(t) = \mathbf{M_h}\vec{u}_h(t);\ \vec{u}_h(0) = \vec{u}_0,$$

with $h = 1/(n+1)$, $\vec{u}_h = (u_h^1, u_h^2, \dots, u_h^n)$, and $\mathbf{M}_h$ being the matrix on the left of (2.3); that is,

$$\mathbf{M}_h := \frac{1}{h^\alpha} \begin{pmatrix} w_1 & w_0 & 0 & \dots & 0 \\ w_2 & w_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & w_1 & w_0 \\ w_n & \cdots & \cdots & w_2 & w_1 \end{pmatrix}. \tag{2.5}$$

Theorem 2.2 *Let $u_0 \in C_0(0,1)$ be such that*

$$u_0(x) = ax^{\alpha-1} + bx^{2\alpha-1} + \int_0^x \frac{(x-s)^\alpha}{\Gamma(\alpha+1)} g(s)\, ds$$

for some $g \in C_0(0,1]$. Then there exists $C > 0$ such that for $t > h^\alpha > 0$,

$$\left|[S(t)u_0](x_i) - [\mathbf{S}_h(t)\vec{u}_0]_i\right| \le Ch^\alpha(1 + \log(t/h^\alpha))(\|u_0\|_\infty + |b| + \|g\|_\infty),$$

for all $1 \le i \le n$, where $\vec{u}_0 = (u_0(x_1), \dots, u_0(x_n))$.

Proof See Section 4. □

The fact that $\mathbf{M}_h$ and A_h^α generate positive contraction semigroups follows from the following proposition showing that $\mathbf{M}_h$ is a 'Q-matrix'.

Proposition 2.1 *Consider the weights w_j on the left hand side of (2.4). Then $w_1 < 0$, $w_j > 0$ for all $j \neq 1$ and $\sum_{j=0}^{\infty} w_j = 0$.*

Proof Taking the Laplace transform of the convolution in the infinite version of (2.4) yields

$$\left(\sum_{k=0}^{\infty} w_k e^{-\lambda k}\right)\left(e^{\lambda}\sum_{j=1}^{\infty} j^{\alpha-1} e^{-j\lambda}\right) = \Gamma(\alpha)$$

or

$$\begin{aligned}\sum_{k=0}^{\infty} w_k e^{-\lambda k} &= \Gamma(\alpha)\Big/\left(e^{\lambda}\sum_{j=1}^{\infty} j^{\alpha-1} e^{-j\lambda}\right)\\ &= \frac{e^{-\lambda}\Gamma(\alpha)}{\mathrm{Li}_{1-\alpha}(e^{-\lambda})},\end{aligned} \tag{2.6}$$

with

$$\mathrm{Li}_s(z) := \sum_{j=1}^{\infty} j^{-s} z^j$$

being the *Polylogarithm* function. It is well known that the polylogarithm for fixed s is analytic in $\mathbb{C}\setminus[1,\infty)$ and is real on $\mathbb{R}\setminus[1,\infty)$. It has the expansion for $|\mu| < 2\pi$ and $s \neq 1, 2, \ldots$ given by

$$\mathrm{Li}_s(e^{-\mu}) = \Gamma(1-s)\mu^{s-1} + \sum_{j=0}^{\infty} \frac{\zeta(s-j)(-1)^j}{j!}\mu^j, \tag{2.7}$$

where ζ is the Riemann Zeta function (see e.g. [5, Eq. (13)], also in Equation (9.3) in [6]). For large moduli; i.e., for $r \to \infty$ and for $-\pi < \theta < \pi$, if $s \neq -1, -2, \ldots$ ([6, Eq. (11.3)])

$$\mathrm{Li}_s(-re^{i\theta}) = -\frac{(i\theta + \ln r)^s}{\Gamma(s+1)} + O((\ln r)^{s-2}). \tag{2.8}$$

Furthermore, along the branch cut for $x > 1$ ([6, Eq. (3.1)]),

$$\lim_{\epsilon\to 0} \mathrm{Im}(\mathrm{Li}_{1-\alpha}(x \pm i\epsilon)) = \pm\frac{\pi(\ln x)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{2.9}$$

Finally, for $-1 < s < 0$, Li_s has a single root at 0 by the Argument Principle along the keyhole contour γ of Li. This is due to the fact that $\mathrm{Li}_{1-\alpha}(\gamma)\cap\mathbb{R}^+ = \mathrm{Li}_{1-\alpha}(\gamma\cap\mathbb{R}^+)$ which follows from the asymptotic behaviours of the polylogarithmic function near the branch cut and near infinity. This in turn implies that the winding number of $\mathrm{Li}_{1-\alpha}(\gamma)$ is one (see also [7, Page 1] and [8, Theorem 4]).

Letting $t = e^{-\lambda}$, equation (2.6) turns into

$$\sum_{k=0}^{\infty} w_k t^k = \frac{t\Gamma(\alpha)}{\sum_{j=1}^{\infty} j^{\alpha-1} t^j} = \frac{t\Gamma(\alpha)}{\mathrm{Li}_{1-\alpha}(t)},$$

which is analytic in $\mathbb{C} \setminus [1, \infty)$ as $\mathrm{Li}_{1-\alpha}$ has only a single root at zero. Hence by the Residue Theorem

$$w_n = \frac{1}{2\pi i} \int_\gamma \frac{\Gamma(\alpha)}{z^n \mathrm{Li}_{1-\alpha}(z)}\, dz,$$

where γ is any closed counter-clockwise simple curve around the origin. We use a keyhole contour for $n \geq 2$, evading $[1, \infty)$. By (2.8) and the expansion for small z, the integrals over the outer and inner arc converge to zero. By (2.9) this leaves

$$\begin{aligned} w_n &= \lim_{\epsilon \to 0} \frac{\Gamma(\alpha)}{2\pi i} \int_1^\infty \frac{1}{x^n \mathrm{Li}_{1-\alpha}(x + i\epsilon)} - \frac{1}{x^n \mathrm{Li}_{1-\alpha}(x - i\epsilon)}\, dx \\ &= \frac{\Gamma(\alpha)}{2\pi i} \int_1^\infty \frac{1}{x^n (\mathrm{Re}(\mathrm{Li}_{1-\alpha}(x)) + i\pi (\ln x)^{-\alpha}/\Gamma(1-\alpha))} \\ &\qquad - \frac{1}{x^n (\mathrm{Re}(\mathrm{Li}_{1-\alpha}(x)) - i\pi (\ln x)^{-\alpha}/\Gamma(1-\alpha))}\, dx \\ &= -\frac{\Gamma(\alpha)}{\Gamma(1-\alpha)} \int_1^\infty \frac{(\ln x)^{-\alpha}}{x^n |\mathrm{Li}_{1-\alpha}(x)|^2}\, dx > 0. \end{aligned}$$

Finally, taking $t \to 1$ (or $\lambda \to 0$),

$$\sum_{k=0}^\infty w_k = \lim_{t \to 1} \frac{t\Gamma(\alpha)}{\mathrm{Li}_{1-\alpha}(t)} = 0,$$

and the proof is complete. □

3 Convergence on an infinite domain

We follow [9] and first show convergence of the scheme on $L^1(\mathbb{R})$. This will then imply convergence on all spaces where the shift-group is strongly continuous (even more general, the appropriately adapted scheme will converge to the α power of the generator of a strongly continuous group).

In order to introduce some notation define the Fourier transform for a function in $L^1(\mathbb{R})$ via

$$\hat{f}(k) = \int e^{-ikx} f(x)\, dx.$$

Let

$$X_\alpha := \{f \in L^1(\mathbb{R}) : \exists f^{(\alpha)} \in L^1(\mathbb{R}) : \widehat{f^{(\alpha)}}(k) = (ik)^\alpha \hat{f}(k)\}$$

and for $f \in X_\alpha$ define

$$\|f\|_\alpha := \left\| f^{(\alpha)} \right\|_{L^1(\mathbb{R})}.$$

Theorem 3.1 *Let $1 < \alpha \leq 2$ and $0 \leq \beta \leq \alpha$. Then there exists $C > 0$ such that for $f \in X_{\alpha+\beta}$ with $A_h^\alpha f$ defined via (1.2),*

$$\left\| A_h^\alpha f - f^{(\alpha)} \right\|_{L^1(\mathbb{R})} \leq C h^\beta \|f\|_{\alpha+\beta}.$$

In case of $\beta = 0$ and $f \in X_\alpha$, $\left\| A_h^\alpha f - f^{(\alpha)} \right\|_{L^1(\mathbb{R})} \to 0.$

Proof The Fourier transform of our approximation scheme is given by

$$\begin{aligned}\widehat{A_h^\alpha f}(k) &= \frac{1}{h^\alpha}\sum_{n=-1}^{\infty} w_{n+1} e^{-iknh}\hat{f}(k) = \frac{\Gamma(\alpha)}{h^\alpha \operatorname{Li}_{1-\alpha}(e^{-ihk})}\hat{f}(k)\\ &= (ik)^\alpha \hat{g}(kh)\hat{f}(k)\\ &= (ik)^\alpha \hat{f}(k) + h^\beta \frac{\hat{g}(kh)-1}{(ikh)^\beta}(ik)^{\alpha+\beta}\hat{f}(k)\end{aligned}$$

with

$$\hat{g}(k) = \frac{\Gamma(\alpha)}{(ik)^\alpha \operatorname{Li}_{1-\alpha}(e^{-ik})}.$$

We claim that

$$\hat{g}_\beta(k) := (\hat{g}(k)-1)/(ik)^\beta \tag{3.10}$$

is the Fourier transform of an L^1-function for all $0<\beta\le\alpha$. This would prove the the theorem for all $0<\beta\le\alpha$ as the L^1 norm of g_β is the same as the L^1-norm of $x\mapsto g_\beta(x/h)/h$. But first we show that $\hat{g}$ is the Fourier transform of an L^1 function with $\hat{g}(0)=1$, which obviously implies the assertion of the theorem in case of $\beta=0$.

We are going to use a variant of the Carlson-Beurling Inequality (see [9, Prop. 2.1])

$$\|f\|_{L^1} \le C(r)\|\hat{f}\|_{L^r}^{\frac{1}{s}}\|\hat{f}'\|_{L^r}^{\frac{1}{r}}, \tag{3.11}$$

where $1<r\le 2$ and $1/r+1/s=1$. Recall that by (2.7)

$$\operatorname{Li}_s(e^{-ik}) = (ik)^{s-1}\Gamma(1-s)+\zeta(s)+O(k),$$

for $|k|\le\pi$. Hence $\hat{g}(0)=1$ and as $|\operatorname{Li}_{1-\alpha}(e^{-ik})|$ is bounded below, $\hat{g}\in L^2(\mathbb{R})$. Furthermore,

$$\frac{d}{dk}\hat{g}(k) = -i\frac{\alpha\Gamma(\alpha)}{(ik)^{\alpha+1}\operatorname{Li}_{1-\alpha}(e^{-ik})} - \frac{\Gamma(\alpha)}{(ik)^\alpha \operatorname{Li}_{1-\alpha}^2(e^{-ik})}(-i\operatorname{Li}_{-\alpha}(e^{-ik})) \tag{3.12}$$

$$\begin{aligned}&= -i\hat{g}\left(e^{-ik}\right)\left(\frac{\alpha}{ik} - \frac{\operatorname{Li}_{-\alpha}(e^{-ik})}{\operatorname{Li}_{1-\alpha}(e^{-ik})}\right)\\ &= -i\hat{g}\left(e^{-ik}\right)\frac{\alpha\operatorname{Li}_{1-\alpha}(e^{-ik}) - ik\operatorname{Li}_{-\alpha}(e^{-ik})}{ik\operatorname{Li}_{1-\alpha}(e^{-ik})}\\ &= -i\hat{g}\left(e^{-ik}\right)\frac{\alpha(ik)^{\alpha-1}\operatorname{Li}_{1-\alpha}(e^{-ik}) - (ik)^\alpha\operatorname{Li}_{-\alpha}(e^{-ik})}{(ik)^\alpha\operatorname{Li}_{1-\alpha}(e^{-ik})} = O(k^{\alpha-1}).\end{aligned} \tag{3.13}$$

Note that in (3.12), the periodic function $\operatorname{Li}_{-\alpha}(e^{-ik})/\operatorname{Li}_{1-\alpha}^2(e^{-ik})$ is bounded as at the singularity $\lim_{k\to 0}\operatorname{Li}_{-\alpha}(e^{-ik})/\operatorname{Li}_{1-\alpha}^2(e^{-ik}) = \lim_{k\to 0}(ik)^{-\alpha-1}/(ik)^{-2\alpha} = 0$. Equation (3.13) shows that $d\hat{g}/dk(0)=0$ and combined with the decay at infinity given by (3.12), $d\hat{g}/dk\in L^2(\mathbb{R})$ as well. Hence $g\in L^1(\mathbb{R})$.

In order to show that g_β is in L^1, note that for $|k|<\pi$,

$$\hat{g}_\beta(k) = \frac{\hat{g}(k)-1}{(ik)^\beta} = (ik)^{\alpha-\beta}(\zeta(1-\alpha)+O(k))/\Gamma(\alpha),$$

and that for large $|k|$, $|\hat{g}_\beta(k)|$ decays like $|k|^{-\beta}$; i.e., there exists $M>0$ such that

$$\left|\hat{g}_\beta(k)\right| \le M|k|^{-\beta}.$$

As to its derivative,

$$\frac{d}{dk}\frac{\hat{g}(k)-1}{(ik)^{\beta}}=i\frac{k\hat{g}'(k)-\beta(\hat{g}(k)-1)}{(ik)^{\beta+1}}.$$

So for large $|k|$ (i.e. $|k|\geq 1$) there exists M such that

$$\left|\hat{g}'_{\beta}(k)\right|\leq M|k|^{-\beta-1}.$$

However for small k and $\beta<\alpha$, $|\hat{g}'_{\beta}(k)|\approx|k|^{\alpha-\beta-1}$, whereas (2.7) implies that for $\beta=\alpha$ the function is bounded in zero.

Consider the partition of unity ($j=1,2,\ldots$)

$$\phi_0(k):=\begin{cases}1 & |k|<1\\ 2-|k| & 1\leq|k|\leq 2\\ 0 & \text{else}\end{cases},$$

$$\phi_j(k):=\begin{cases}(|k|-2^{j-1})2^{1-j} & 2^{j-1}\leq|k|\leq 2^j\\ (2^{j+1}-|k|)2^{-j} & 2^j\leq|k|\leq 2^{j+1}\\ 0 & \text{else}\end{cases}.$$

Then by (3.11), $\phi_j\hat{g}_{\beta}$ are the Fourier transforms of L^1 functions and for $j\geq 1$, using the tail estimates of $\hat{g}_{\beta}$ and $\hat{g}'_{\beta}$,

$$\left\|\phi_j\hat{g}_{\beta}\right\|_2^{1/2}\left\|(\phi_j\hat{g}_{\beta})'\right\|_2^{1/2}\leq C2^{-\beta j}.$$

As $\hat{g}_{\beta}=\sum_{j=0}^{\infty}\phi_j\hat{g}_{\beta}$, $g_{\beta}\in L^1(\mathbb{R})$.

□

3.1 Transference to other Banach spaces

In order to transfer Theorem 3.1 we use the unbounded functional calculus developed in [10]. In particular, recall that for $\hat{f}(k)=\int e^{-ikt}f(t)\,dt$ and $-A$ being the generator of a strongly continuous group $(G(t))_{t\in\mathbb{R}}$ on a Banach space X, one may define

$$\hat{f}(-iA):=\int_{\mathbb{R}}f(t)G(t)\,dt,$$

where the integral is understood in the strong Bochner sense.

Theorem 3.2 *Let $-A$ be the generator of a bounded strongly continuous group $(G(t))_{t\in\mathbb{R}}$ on a Banach space X and define*

$$A_h^{\alpha}f:=\frac{1}{h^{\alpha}}\sum_{n=-1}^{\infty}w_{n+1}G(nh)f.$$

Then for $0\leq\beta<\alpha$ and $f\in D(A^{\alpha+\beta})$,

$$\left\|A_h^{\alpha}f-A^{\alpha}f\right\|_X\leq Ch^{\beta}\left\|A^{\alpha+\beta}f\right\|_X.$$

In case $\beta=0$, $\|A_h^{\alpha}f-A^{\alpha}f\|_X\to 0$.

Proof Consider $\hat{g}_\beta$ of (3.10). Then

$$\left\| h^\beta \hat{g}_\beta(-ihA) \right\|_{\mathcal{B}(X)} \leq Ch^\beta$$

and hence $\left\| h^\beta \hat{g}_\beta(-ihA)A^{\alpha+\beta} f \right\|_X \leq Ch^\beta \left\| A^{\alpha+\beta} f \right\|_X$. But

$$h^\beta \hat{g}_\beta(-ihA)A^{\alpha+\beta} f = A_h^\alpha f - A^\alpha f.$$

In case of $\beta = 0$, the strong continuity of G implies that

$$\hat{g}_0(-ihA)f = \int_{\mathbb{R}} G(s) f g(s/h)/h \, ds - f \to 0$$

for all $f \in X$. □

Corollary 3.1 *The approximation scheme converges uniformly at rate h^α in $C_0(\mathbb{R})$ for 2α-times continuously differentiable functions.*

3.2 Uniform analyticity of approximating semigroups

Following [9, Section 4], let

$$\psi(k) := \frac{\Gamma(\alpha)}{\mathrm{Li}_{1-\alpha}(e^{-ik})} = \sum_{j=-1}^{\infty} w_{j+1} e^{-ikj}.$$

Using that $\sum w_j = 0$, $w_1 < 0$ and $w_j > 0$ for all $j \neq 1$, we deduce that $\mathrm{Re}\,\psi(k) < 0$ for $k \neq 0$. As

$$\frac{d}{dk}\psi(k) = i\frac{\Gamma(\alpha)\,\mathrm{Li}_{-\alpha}(e^{-ik})}{(\mathrm{Li}_{1-\alpha}(e^{-ik}))^2} \approx \frac{\Gamma(\alpha)(ik)^{-\alpha-1}}{(ik)^{-2\alpha}\Gamma(\alpha+1)} = \frac{1}{\alpha}(ik)^{\alpha-1},$$

the conditions of [9, Theorem 4.1] are satisfied.

This shows that the Fourier multipliers defined by our approximation generate uniformly analytic semigroups of angle α on $L^1(\mathbb{R})$. Following [9, Section 4], this implies if one employs the backward Euler scheme as time integrator, then the fully discrete scheme converges unconditionally at rate $\Delta t + (\Delta x)^\alpha \log \Delta x$ to the solution of the Cauchy problem on $\mathbb{R}$ for initial conditions $u_0 \in D(A^\alpha)$ in any space where the translation group is strongly continuous, in particular in $L^p(\mathbb{R})$ and $C_0(\mathbb{R})$.

4 Dirichlet problem on bounded domain

In order to prove convergence of order α in space to solutions of (1.1), we first show that the solution semigroup is analytic and that the matrix $\mathbf{M}_h$ governing the approximation scheme can be used to define generators of contraction semigroups on finite dimensional subspaces. We then show that the semigroups converge to each other at the required rate if the initial condition is smooth enough even though in general the generators converge at a much slower rate.

To keep notation consistent we denote the generators in this section also by A^α and A_h^α, but we would like to stress that due to the imposition of boundary conditions these operators are not fractional powers of the first derivative operator anymore.

4.1 Analyticity of solution semigroup

Recall from [1] that the solution semigroup governing (1.1) on $C_0(0,1)$ has generator $(A^\alpha, D(A^\alpha))$ with

$$D(A^\alpha) := \{f : f(x) = I^\alpha g(x) - I^\alpha g(1)x^{\alpha-1},\ g \in C_0(0,1)\} \tag{4.14}$$

and $A^\alpha f = g$ for $f \in D(A^\alpha)$; here $I^\alpha g(x) = \int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\,ds$ is the α-fractional antiderivative of g.

In a weak sense, the solution to

$$\lambda f - A^\alpha f = \delta_y$$

is given for $\operatorname{Re}\lambda \geq 0$ by

$$\begin{aligned} f_y(x) :=& -H(x-y)\sum_{n=1}^{\infty}\frac{(x-y)^{n\alpha-1}}{\Gamma(n\alpha)}\lambda^{n-1} \\ &+ \frac{\sum_{n=1}^{\infty}\frac{(1-y)^{n\alpha-1}}{\Gamma(n\alpha)}\lambda^{n-1}}{\sum_{n=1}^{\infty}\frac{1}{\Gamma(n\alpha)}\lambda^{n-1}}\sum_{n=1}^{\infty}\frac{x^{n\alpha-1}}{\Gamma(n\alpha)}\lambda^{n-1}. \end{aligned}$$

Hence

$$R(\lambda, A^\alpha)g(x) = \int_0^1 g(y) f_y(x)\,dy.$$

The function f_y can be expressed in terms of the Mittag-Leffler function

$$E_{\alpha,0}(z) = \sum_{n=1}^{\infty}\frac{z^n}{\Gamma(n\alpha)}$$

via

$$\begin{aligned} f_y =& -H(x-y)\frac{1}{\lambda(x-y)}E_{\alpha,0}(\lambda(x-y)^\alpha) \\ &+ \frac{1}{\lambda(1-y)x}\frac{E_{\alpha,0}(\lambda(1-y)^\alpha)}{E_{\alpha,0}(\lambda)}E_{\alpha,0}(\lambda x^\alpha). \end{aligned}$$

For $|z|$ large with $\arg z < \alpha\pi/2$ it has the asymptotic expansion (see [11, $\mathbf{S}_1 8$])

$$E_{\alpha,0}(z) = \frac{z^{1/\alpha}}{\alpha}\exp(z^{1/\alpha}) - \frac{1}{z\Gamma(-\alpha)} + O(|z|^{-2}).$$

To show that $\lambda R(\lambda, A^\alpha)$ is bounded we start with an estimate for the Green's function f_y.

Proposition 4.1 *Let $|\lambda|$ be large enough so that $|E_{\alpha,0}(\lambda)| \geq \left|\frac{\lambda^{1/\alpha}}{2\alpha}\exp(\lambda^{1/\alpha})\right|$. Then there exists $M>0$ such that for all $x, y \in (0,1)$,*

$$|f_y(x)| \leq \begin{cases} M\left|\frac{\lambda^{1/\alpha}}{\lambda}\right|\exp(-(y-x)\operatorname{Re}\lambda^{1/\alpha}) & x < y + |\lambda|^{-1/\alpha} \\ M/(|\lambda|^2(x-y)^{1+\alpha}) & \textit{else} \end{cases}.$$

Proof Note that

$$\begin{aligned} |f_y(x)| \leq &\left|\frac{2\alpha\exp(-\lambda^{1/\alpha})}{\lambda^{1+1/\alpha}}\right| \times \\ &\left|\frac{E_{\alpha,0}(\lambda(1-y)^\alpha)}{1-y}\frac{E_{\alpha,0}(\lambda x^\alpha)}{x} - H(x-y)\frac{E_{\alpha,0}(\lambda(x-y)^\alpha)}{x-y}E_{\alpha,0}(\lambda)\right|. \end{aligned}$$

Furthermore, for $|\lambda| r^\alpha < 1$, there exists $M > 0$ such that

$$\left|\frac{E_{\alpha,0}(\lambda r^\alpha)}{r}\right| \leq M|\lambda| r^{\alpha-1} \leq M|\lambda|^{1/\alpha}$$

and for $|\lambda| r^\alpha \geq 1$,

$$\left|\frac{E_{\alpha,0}(\lambda r^\alpha)}{r}\right| = \left|\lambda^{1/\alpha}\exp(\lambda^{1/\alpha}r)/\alpha + h(\lambda r^\alpha)/r\right| \leq |\lambda^{1/\alpha}|\left(|\exp(\lambda^{1/\alpha}r)| + M\right);$$

with h being the tail of the asymptotic expansion satisfying $|h(z)| \leq M/|z|$. In both cases we obtain

$$\left|\frac{E_{\alpha,0}(\lambda r^\alpha)}{r}\right| \leq M|\lambda|^{1/\alpha}\exp(r\operatorname{Re}\lambda^{1/\alpha}), \tag{4.15}$$

proving the estimate for $|f_y(x)|$ in case of $x < y + |\lambda|^{-1/\alpha}$.

In case of $x \geq y + |\lambda|^{-1/\alpha}$,

$$\begin{aligned} |f_y(x)| \leq &\left|\frac{2\alpha\exp(-\lambda^{1/\alpha})}{\lambda^{1+1/\alpha}}\right| \times \\ &\left|\frac{h(\lambda(1-y)^\alpha)}{1-y}\frac{\lambda^{1/\alpha}}{\alpha}\exp(\lambda^{1/\alpha}x) + \frac{h(\lambda x^\alpha)}{x}\frac{\lambda^{1/\alpha}}{\alpha}\exp(\lambda^{1/\alpha}(1-y))\right. \\ &+\frac{h(\lambda(1-y)^\alpha)}{1-y}\frac{h(\lambda x^\alpha)}{x} - \frac{h(\lambda(x-y)^\alpha)}{x-y}\frac{\lambda^{1/\alpha}}{\alpha}\exp(\lambda^{1/\alpha}) \\ &\left. - h(\lambda)\frac{\lambda^{1/\alpha}}{\alpha}\exp(\lambda^{1/\alpha}(x-y)) - \frac{h(\lambda(x-y)^\alpha)}{x-y}h(\lambda)\right| \\ \leq &\frac{M}{|\lambda^2|(x-y)^{1+\alpha}}. \end{aligned}$$

□

Theorem 4.1 *Let $(A^\alpha, D(A^\alpha))$ be the generator of the semigroup given by the fractional derivative operator of order $1 < \alpha \le 2$ on $C_0(0,1)$ or $L^1[0,1]$ with Dirichlet boundary conditions. Then there exists $M > 0$ such that for all $\operatorname{Re}\lambda \ge 0$,*

$$\left\|R(\lambda, A^\alpha)\right\| \le M/|\lambda|;$$

i.e., the semigroup is analytic.

Proof Note that from above,

$$R(\lambda, A^\alpha)g(x) = \int_0^1 f_y(x)g(y)\,dy.$$

Hence, by Proposition 4.1, if the underlying Banach space is $X = C_0(0,1)$,

$$\begin{aligned}
\left\|R(\lambda, A^\alpha)\right\| &= \sup_{\|g\|=1}\left\|R(\lambda, A^\alpha)g\right\| = \sup_{\|g\|=1}\sup_{x\in(0,1)}\left|\int_0^1 f_y(x)g(y)\,dy\right| \\
&\le \sup_{x\in(0,1)}\int_0^1 |f_y(x)|\,dy \\
&= \sup_{x\in(0,1)}\int_0^{x-|\lambda|^{-1/\alpha}\vee 0} |f_y(x)|\,dy + \int_{x-|\lambda|^{-1/\alpha}\vee 0}^1 |f_y(x)|\,dy \\
&\le \sup_{x\in(0,1)}\frac{M}{|\lambda|^2}\int_0^{x-|\lambda|^{-1/\alpha}\vee 0}\frac{1}{(x-y)^{1+\alpha}}\,dy \\
&\quad + \frac{M|\lambda|^{1/\alpha}}{|\lambda|}\int_{x-|\lambda|^{-1/\alpha}\vee 0}^1 \exp(-(y-x)\operatorname{Re}\lambda^{1/\alpha})\,dy \\
&\le \frac{M}{\alpha|\lambda|^2}|\lambda| + \frac{M}{|\lambda|}\frac{|\lambda|^{1/\alpha}}{\operatorname{Re}\lambda^{1/\alpha}}\exp\left(|\lambda|^{-1/\alpha}\operatorname{Re}\lambda^{1/\alpha}\right) \\
&= \frac{M}{\alpha|\lambda|} + \frac{M}{|\lambda|}\frac{\exp(\cos(\pi/2\alpha))}{\cos(\pi/2\alpha)}.
\end{aligned}$$

Similarly for $X = L^1[0,1]$, one has

$$\left\|R(\lambda, A^\alpha)\right\| \le \sup_{y\in[0,1]}\int_0^1 |f_y(x)|\,dx \le M/|\lambda|.$$

□

4.2 Piece-wise power interpolation

As the rate of convergence hinges on mapping the function $x^{\alpha-1}$ exactly, we introduce a *piece-wise power interpolation.* For an n-dimensional vector $y = (y_1, \ldots, y_n)$ define

$$(E_n y)(x) := \frac{y_{i+1} - y_i}{x_{i+1}^{\alpha-1} - x_i^{\alpha-1}}x^{\alpha-1} + \frac{x_{i+1}^{\alpha-1}y_i - x_i^{\alpha-1}y_{i+1}}{x_{i+1}^{\alpha-1} - x_i^{\alpha-1}},$$

where $ih = x_i \le x \le x_{i+1} = (i+1)h$ and $(n+1)h = 1$ and $y_0 = y_{n+1} = 0$.

Let $\Pi_n : C(0,1) \to C(0,1)$ be the projection to the n-dimensional power interpolation subspace V_n via

$$\Pi_n f := E_n(f(x_i)_{i=1,\ldots,n}),$$

where

$$V_n = \{\Pi_n f : f \in C_0(0,1)\} \subset C_0(0,1).$$

Note that $\Pi_n f(x_i) = f(x_i)$ for all $x_i = ih$, $i = 0,\ldots,n+1$. Furthermore, for continuous functions in $C_0(0,1]$ we extend the projection in the obvious way. In particular, $\Pi_n(ax^{\alpha-1}) = ax^{\alpha-1}$.

Define

$$A_h^\alpha f := E_n \mathbf{M}_h(f(x_i))_{i=1,\ldots,n},$$

where $\mathbf{M}_h$ is given by (2.5). By mapping the values on the grid points $x_i, 1 \leq i \leq n$ and the properties of the weights given in Proposition 2.1; i.e. $\mathbf{M}_h$ is a so-called Q-matrix, the operator A_h^α is the generator of a contraction semigroup $\{S_h(t)\}_{t\geq 0}$ on the n-dimensional power interpolation subspace V_n and is a bounded linear operator on $C_0(0,1)$. For $f \in C_0(0,1]$ we extend A_h^α by increasing the dimension of $\mathbf{M}_h$ by one and setting the last row equal to zero. In particular $A_h^\alpha x^{\alpha-1} = 0$.

Proposition 4.2 *Let $(A^\alpha, D(A^\alpha))$ be the generator of the semigroup given by the fractional derivative operator of order $1 < \alpha \leq 2$ on $C_0(0,1)$ and let $f \in D(A^\alpha)$ and $(n+1)h = 1$. Then there exists $C > 0$ such that*

$$\|\Pi_n f - f\| \leq C h^\alpha \|A^\alpha f\|.$$

Proof Let $f \in D(A^\alpha)$ and $x = x_i + \lambda h$ for some $x_i = ih$ and $0 \leq \lambda \leq 1$. Then by (4.14), $f = f_0 - f_0(1)x^{\alpha-1}$ with $f_0(x) = \int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\,ds$ for some $g \in C_0(0,1)$. As the power interpolation is exact for $x^{\alpha-1}$, it follows that

$$\begin{aligned}\Pi_n f(x) - f(x) =& \frac{x^{\alpha-1} - x_i^{\alpha-1}}{(x_i+h)^{\alpha-1} - x_i^{\alpha-1}} f_0(x_i+h) + \frac{(x_i+h)^{\alpha-1} - x^{\alpha-1}}{(x_i+h)^{\alpha-1} - x_i^{\alpha-1}} f_0(x_i)\\ &- f_0(x)\\ =& \frac{x^{\alpha-1} - x_i^{\alpha-1}}{(x_i+h)^{\alpha-1} - x_i^{\alpha-1}} (f_0(x_i+h) - f_0(x))\\ &+ \frac{(x_i+h)^{\alpha-1} - x^{\alpha-1}}{(x_i+h)^{\alpha-1} - x_i^{\alpha-1}} (f_0(x_i) - f_0(x)).\end{aligned}$$

Furthermore,

$$\begin{aligned}|f_0(x_i+h) &- f_0(x) - (1-\lambda)h f_0'(x)|\\ &= \left| \int_0^{x_i+h} \frac{(x_i+h-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\,ds - \int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\,ds \right.\\ &\quad \left. -(1-\lambda)h \int_0^x \frac{(x-s)^{\alpha-2}}{\Gamma(\alpha-1)} g(s)\,ds \right|\end{aligned}$$

$$
\begin{aligned}
&= \bigg| \int_0^x \frac{(x_i+h-s)^{\alpha-1} - (x-s)^{\alpha-1} - (\alpha-1)(1-\lambda)h(x-s)^{\alpha-2}}{\Gamma(\alpha)} g(s)\, ds \\
&\quad + \int_x^{x_i+h} \frac{(x_i+h-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\, ds \bigg| \\
&\le \|g\| \left| \frac{(x_i+h-x)^\alpha - (x_i+h)^\alpha + x^\alpha + \alpha(1-\lambda)hx^{\alpha-1}}{\Gamma(\alpha+1)} \right| \\
&\quad + \|g\| \frac{(x_i+h-x)^\alpha}{\Gamma(\alpha+1)} \\
&\le \|g\| \left(2\frac{(1-\lambda)^\alpha h^\alpha}{\Gamma(\alpha+1)} + \left| \frac{(x+(1-\lambda)h)^\alpha - x^\alpha - \alpha(1-\lambda)hx^{\alpha-1}}{\Gamma(\alpha+1)} \right| \right).
\end{aligned}
$$

For $x \ge h$ the right hand side is maximal at $x = h$ and hence

$$
\left| f_0(x_i+h) - f_0(x) - (1-\lambda)hf_0'(x) \right| \le Ch^\alpha.
$$

Similarly,

$$
\left| f_0(x) - f_0(x_i) - \lambda h f_0'(x_i) \right| \le Ch^\alpha.
$$

Hence

$$
\begin{aligned}
\Pi_n f(x) - f(x) =& \frac{x^{\alpha-1} - x_i^{\alpha-1}}{(x_i+h)^{\alpha-1} - x_i^{\alpha-1}} (1-\lambda)hf_0'(x) \\
&- \frac{(x_i+h)^{\alpha-1} - x^{\alpha-1}}{(x_i+h)^{\alpha-1} - x_i^{\alpha-1}} \lambda h f_0'(x_i) + O(h^\alpha).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\frac{x^{\alpha-1} - x_i^{\alpha-1}}{(x_i+h)^{\alpha-1} - x_i^{\alpha-1}} &= \frac{(\alpha-1)\lambda h x_i^{\alpha-2} + O(h^2 x_i^{\alpha-3})}{(x_i+h)^{\alpha-1} - x_i^{\alpha-1}} \\
&= \frac{\lambda + O(h/x_i)}{(x_i+h)^{\alpha-1} - x_i^{\alpha-1})/((\alpha-1)hx_i^{\alpha-2})},
\end{aligned}
$$

and as the denominator is $1 + O(h/x_i)$,

$$
\frac{(x_i+h)^{\alpha-1} - x^{\alpha-1}}{(x_i+h)^{\alpha-1} - x_i^{\alpha-1}} = \frac{1 - \lambda + O(h/x_i)}{(x_i+h)^{\alpha-1} - x_i^{\alpha-1})/((\alpha-1)hx_i^{\alpha-2})}
$$

and therefore

$$
\begin{aligned}
\Pi_n f(x) - f(x) =& \frac{\lambda(1-\lambda)h}{((x_i+h)^{\alpha-1} - x_i^{\alpha-1})/(\alpha-1)hx_i^{\alpha-2}} \left(f_0'(x) - f_0'(x_i) \right) \\
&+ (1-\lambda)hO(h/x_i)f_0'(x) + \lambda h O(h/x_i) f_0'(x_i).
\end{aligned}
$$

Since $|f_0'(x) - f_0'(x_i)| \le \|g\| h^{\alpha-1}/\Gamma(\alpha)$ and $|f_0'(x)| \le \|g\| x^{\alpha-1}/\Gamma(\alpha)$, it follows that

$$
\begin{aligned}
|\Pi_n f(x) - f(x)| \le& \frac{\lambda(1-\lambda)h}{(2^{\alpha-1}-1)/(\alpha-1)} h^{\alpha-1} \|g\|/\Gamma(\alpha) \\
&+ Ch^2 \left(\frac{(x_i+h)^{\alpha-1}}{x_i} + \frac{x_i^{\alpha-1}}{x_i} \right) \\
\le& Ch^\alpha \|g\| = Ch^\alpha \|A^\alpha f\|,
\end{aligned}
$$

as decreasing x_i increases the right hand side; i.e., the right hand side is maximal if $x_i = h$.

In case $x < h$, $x_i = 0$ and hence

$$
|\Pi_n f(x) - f(x)| = \frac{x^{\alpha-1}}{h^{\alpha-1}} |f_0(h)| \le \frac{h^\alpha}{\Gamma(\alpha+1)} \|g\| = \frac{h^\alpha}{\Gamma(\alpha+1)} \|A^\alpha f\|
$$

as well. □

Remark 4.1 Note that in the above proof the fact that $g(1) = 0$ is not needed.

Proposition 4.3 *Let $g \in C_0(0,1)$. Assume there exists $C_g > 0$ such that for all n, j with $0 \le j \le n-1$ and $h = 1/(n+1), x_j = jh$,*

$$\left| \frac{g(x_{j+1}) - g(x_j)}{x_{j+1}^{\alpha-1} - x_j^{\alpha-1}} \right| < C_g.$$

Then there exists $C > 0$ independent of g such that

$$\| \left(A^\alpha\right)^{-1} \Pi_n g - \left(A_h^\alpha\right)^{-1} \Pi_n g\| \le C h^\alpha (C_g + \|g\|).$$

Proof Note that for $\lambda = 0$ the formula for the negative resolvent reads as

$$\left(A^\alpha\right)^{-1} g(x) = \int_0^x \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)} g(y)\, dy - x^{\alpha-1} \int_0^1 \frac{(1-y)^{\alpha-1}}{\Gamma(\alpha)} g(y)\, dy,$$

which implies that

$$\begin{aligned}
&\left(A^\alpha\right)^{-1} \Pi_n g(x_i) = \\
&\qquad \sum_{j=0}^{i-1} \int_{x_j}^{x_{j+1}} \frac{(x_i - y)^{\alpha-1}}{\Gamma(\alpha)} \left[\frac{y^{\alpha-1} - x_j^{\alpha-1}}{x_{j+1}^{\alpha-1} - x_j^{\alpha-1}} (g(x_{j+1}) - g(x_j)) + g(x_j) \right] dy \\
&\quad - x_i^{\alpha-1} \sum_{j=0}^{n} \int_{x_j}^{x_{j+1}} \frac{(1-y)^{\alpha-1}}{\Gamma(\alpha)} \left[\frac{y^{\alpha-1} - x_j^{\alpha-1}}{x_{j+1}^{\alpha-1} - x_j^{\alpha-1}} (g(x_{j+1}) - g(x_j)) + g(x_j) \right] dy.
\end{aligned}$$

Let $(n+1)h = 1$, $x_j = jh$; $j = 1, \ldots, n$, and $\mathbf{M}_h$ be defined by (2.5). Then

$$\left(\mathbf{M}_h^{-1} e_j\right)_i = h \left(H(i-j) \frac{(ih - jh)^{\alpha-1}}{\Gamma(\alpha)} - (hi)^{\alpha-1} \frac{(1-jh)^{\alpha-1}}{\Gamma(\alpha)} \right),$$

where H is the Heaviside step function. Hence

$$\begin{aligned}
\left(A_h^\alpha\right)^{-1} \Pi_n g(x_i) &= \sum_{j=1}^{i} h \frac{(x_i - x_j)^{\alpha-1}}{\Gamma(\alpha)} g(x_j) - x_i^{\alpha-1} \sum_{j=1}^{n+1} h \frac{(1-x_j)^{\alpha-1}}{\Gamma(\alpha)} g(x_j) \\
&= \frac{h}{2} \left(\sum_{j=0}^{i-1} \frac{(x_i - x_{j+1})^{\alpha-1}}{\Gamma(\alpha)} g(x_{j+1}) + \sum_{j=0}^{i-1} \frac{(x_i - x_j)^{\alpha-1}}{\Gamma(\alpha)} g(x_j) \right) \\
&\quad - x_i^{\alpha-1} \frac{h}{2} \left(\sum_{j=0}^{n} \frac{(1-x_{j+1})^{\alpha-1}}{\Gamma(\alpha)} g(x_{j+1}) + \sum_{j=0}^{n} \frac{(1-x_j)^{\alpha-1}}{\Gamma(\alpha)} g(x_j) \right),
\end{aligned}$$

using that $g(0) = 0$. Note that $(A_h^\alpha)^{-1} \Pi_n g(x_i)$ is the trapezoidal rule approximation of $(A^\alpha)^{-1} \Pi_n g(x_i)$ and that the second integrals are a special case of the first integrals using $x_i = 1$.

Comparing each summand of $(A^\alpha)^{-1} \Pi_n g(x_i) - (A_h^\alpha)^{-1} g(x_i)$ for $x_i \ge x_{j+1}$ and $i \le n+1$, the trapezoidal approximation yields

$$\frac{g(x_{j+1}) - g(x_j)}{x_{j+1}^{\alpha-1} - x_j^{\alpha-1}} \left(\int_{x_j}^{x_{j+1}} \frac{(x_i - y)^{\alpha-1}}{\Gamma(\alpha)} (y^{\alpha-1} - x_j^{\alpha-1})\, dy \right.$$

$$
- \frac{h}{2}\frac{(x_i - x_{j+1})^{\alpha-1}(x_{j+1}^{\alpha-1} - x_j^{\alpha-1})}{\Gamma(\alpha)}\Bigg)
$$

$$
= \frac{g(x_{j+1}) - g(x_j)}{x_{j+1}^{\alpha-1} - x_j^{\alpha-1}} h^3 \frac{f''(\xi_j)}{12},
$$

and

$$
g(x_j)\left(\int_{x_j}^{x_{j+1}} \frac{(x_i - y)^{\alpha-1}}{\Gamma(\alpha)}\,dy - \frac{h}{2}\frac{(x_i - x_{j+1})^{\alpha-1} + (x_i - x_j)^{\alpha-1}}{\Gamma(\alpha)}\right)
$$

$$
= - h^3 g(x_j)\frac{(x_i - \xi_j)^{\alpha-3}}{12\Gamma(\alpha-2)}
$$

for some $\xi_j \in [x_j, x_{j+1}]$ and $f(y) = \frac{(x_i-y)^{\alpha-1}}{\Gamma(\alpha)}(y^{\alpha-1} - x_j^{\alpha-1})$.

Note that for $0 < j < i - 1$,

$$
h^3(x_i - \xi_j)^{\alpha-3} \leq h^\alpha (i - j - 1)^{\alpha-3},
$$

$$
\begin{aligned}
h^3(x_i - \xi_j)^{\alpha-2}\xi_j^{\alpha-2} &\leq h^3(x_i - x_{j+1})^{\alpha-2}x_j^{\alpha-2} \\
&= h^2(hi)^{2\alpha-3}\left(\frac{1}{i}\left(1 - \frac{j+1}{i}\right)^{\alpha-2}\left(\frac{j}{i}\right)^{\alpha-2}\right) \\
&\leq \max\{h^{2\alpha-1}, h^2\}\left(\frac{1}{i}\left(1 - \frac{j+1}{i}\right)^{\alpha-2}\left(\frac{j}{i}\right)^{\alpha-2}\right),
\end{aligned}
$$

and

$$
h^3(x_i - \xi_j)^{\alpha-1}\xi^{\alpha-3} \leq h^\alpha j^{\alpha-3}.
$$

As

$$
\sum_{j=1}^{i-2}(i - j - 1)^{\alpha-3} \leq \sum_{j=1}^{\infty} j^{\alpha-3} = \zeta(3 - \alpha) < \infty
$$

and

$$
\sum_{j=1}^{i-2}\frac{1}{i}\left(1 - \frac{j+1}{i}\right)^{\alpha-2}\left(\frac{j}{i}\right)^{\alpha-2} \to \int_0^1 (1 - y)^{\alpha-2}y^{\alpha-2}\,dy,
$$

the sums are bounded independent of i. Since for $j = 0$ or $j = i - 1$ the error is also of order h^α, the theorem follows. □

The next lemma sets the scene for the regularity needed of the initial condition to facilitate higher order convergence of our finite difference scheme. We require regularity of order $\alpha + 1$ except for the two terms containing $x^{\alpha-1}$ and $x^{2\alpha-1}$ usually found as part of elements in $D((A^\alpha)^\infty)$ ensuring that if, for example, $f \in D((A^\alpha)^2)$, then f will have the required regularity.

Lemma 4.1 *Let*

$$
f(x) = ax^{\alpha-1} + bx^{2\alpha-1} + \int_0^x \frac{(x - s)^\alpha}{\Gamma(\alpha+1)}g(s)\,ds
$$

for some $g \in C(0, 1]$ and assume $f(1) = 0$. Then $A_h^\alpha \Pi_n f$ satisfies the conditions of Proposition 4.3 with

$$
C_{A_h^\alpha \Pi_n f} \leq C(|b| + \|g\|).
$$

Furthermore,

$$\|A_h^\alpha \Pi_n f\| \le C(|b| + \|\int g\|).$$

Proof By design, $A_h^\alpha \Pi_n x^{\alpha-1} = 0$. Consider g_α of equation (3.10). Then

$$A_h^\alpha \Pi_n \frac{x^{2\alpha-1}}{\Gamma(2\alpha)}(x_i) = \frac{x_i^{\alpha-1}}{\Gamma(\alpha)} + h^\alpha g_\alpha\left(\frac{x_i}{h}\right)/h = \frac{x_i^{\alpha-1}}{\Gamma(\alpha)} + h^{\alpha-1} g_\alpha(i).$$

Since $g_\alpha \in L^1$ and g_α is α times differentiable, its derivative is also a bounded continuous L^1 function. As any function with a bounded continuous derivative satisfies the conditions of Proposition 4.3 so does $A_h^\alpha \Pi_n x^{2\alpha-1}$ and the last term of f.

Finally, by Theorem 2.1, $A_h^\alpha \Pi_n f \to b\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} x^{\alpha-1} + \int_0^x g(s)\,ds$ uniformly on grid points and therefore

$$\|A_h^\alpha \Pi_n f\| \le C(|b| + \|\int g\|).$$

□

Theorem 4.2 *Let*

$$f(x) = ax^{\alpha-1} + bx^{2\alpha-1} + \int_0^x \frac{(x-s)^\alpha}{\Gamma(\alpha+1)} g(s)\,ds$$

with $f(1) = 0$ and some $g \in C(0,1]$ be the initial condition to (1.1). Then, for $t > h^\alpha$, there exists a constant $C > 0$ such that

$$\|S(t)f - S_h(t)\Pi_n f\| \le Ch^\alpha(1 + \log(t/h^\alpha))(\|f\| + |b| + \|g\|).$$

Proof First note that both semigroups are contraction semigroups. Hence

$$\|S(t)f - S(t)\Pi_n f\| \le C\|f - \Pi_n f\| \le Ch^\alpha \|A^\alpha f\| \le Ch^\alpha(|b| + \|\int g\|).$$

Next rewrite

$$\begin{aligned} S(t)\Pi_n f - S_h(t)\Pi_n f &= \int_0^t S(r)(A^\alpha - A_h^\alpha) S_h(t-r)\Pi_n f\,dr \\ &= \int_0^{h^\alpha} S(r)(A^\alpha - A_h^\alpha) S_h(t-r)\Pi_n f\,dr \\ &\quad + \int_{h^\alpha}^t A^\alpha S(r)\big((A_h^\alpha)^{-1} - (A^\alpha)^{-1}\big) A_h^\alpha S_h(t-r)\Pi_n f\,dr \\ &= e_1 + e_2. \end{aligned}$$

Integration by parts then yields

$$\begin{aligned} e_1 &= \int_0^{h^\alpha} S(r)A^\alpha S_h(t-r)\Pi_n f\,dr - \int_0^{h^\alpha} S(r)A_h^\alpha S_h(t-r)\Pi_n f\,dr \\ &= S(h^\alpha)S_h(t-h^\alpha)\Pi_n f - S_h(t)\Pi_n f, \end{aligned}$$

which we further decompose into

$$e_1 = S(h^\alpha)\left[S_h(t-h^\alpha) - S_h(t)\right]\Pi_n f + \left[S(h^\alpha) - I\right]\left(A_h^\alpha\right)^{-1} S_h(t) A_h^\alpha \Pi_n f$$

$$= S(h^\alpha)\left[S_h(t-h^\alpha) - S_h(t)\right]\Pi_n f$$
$$+ \left[S(h^\alpha) - I\right]\left(\left(A_h^\alpha\right)^{-1} - \left(A^\alpha\right)^{-1}\right) S_h(t) A_h^\alpha \Pi_n f$$
$$+ \left[S(h^\alpha) - I\right]\left(A^\alpha\right)^{-1} S_h(t) A_h^\alpha \Pi_n f.$$

Hence

$$\|e_1\| \le h^\alpha \|A_h^\alpha \Pi_n f\| + Ch^\alpha(\|A_h^\alpha \Pi_n f\| + C_{S_h(t)A_h^\alpha \Pi_n f}) + h^\alpha \|A_h^\alpha \Pi_n f\|,$$

where C_g is the constant of Proposition 4.3. As S_h is a contraction,

$$C_{S_h(t)A_h^\alpha \Pi_n f} \le C_{A_h^\alpha \Pi_n f} \le C(|b| + \|g\|).$$

To bound e_2, we use the analyticity of S to get

$$\|e_2\| \le \int_{h^\alpha}^t \frac{C}{r} \|(\left(A_h^\alpha\right)^{-1} - \left(A^\alpha\right)^{-1}) S_h(t-r) A_h^\alpha \Pi_n f\|\, dr$$
$$\le Ch^\alpha \log(t/h^\alpha)(\|A_h^\alpha \Pi_n f\| + C_{A_h^\alpha \Pi_n f}).$$

As $\|A_h^\alpha \Pi_n f\| \le C(|b| + \|\int g\|)$ and $\|\int g\| \le \|g\|$, the proof is complete. □

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