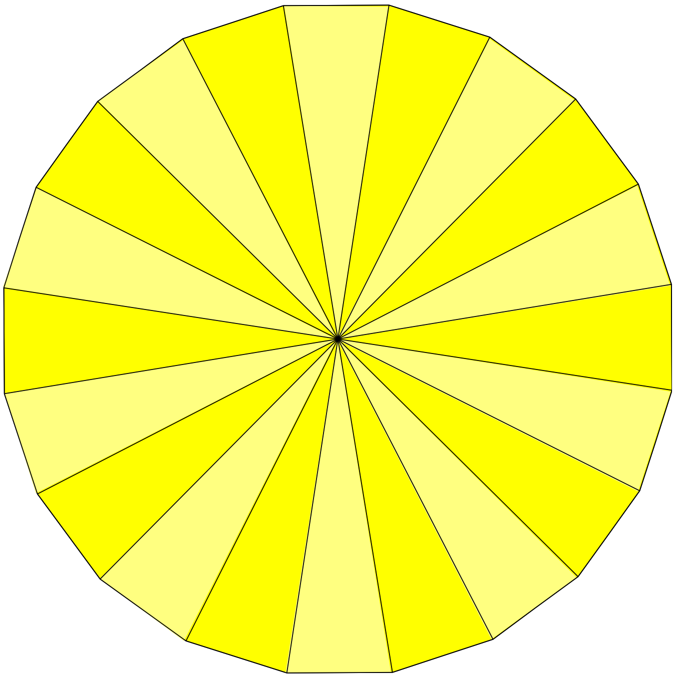
Junior Mathematics Competition

Organised by the Department of Mathematics and Statistics | University of Otago

UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND



2013 Solutions and Comments

Year 9 (Form 3) Prize Winners

First	Matthew Beardsworth	Te Aho o Te Kura Pounamu
Second	Edward Chen	Palmerston North Boys' High School
Third	Jack Craig	Otago Boys' High School

Top 30 (in School Order):

Aiden Burgess, Auckland Grammar School
Bon-Nyeong Goo, Auckland Grammar School
Daniel Ji, Auckland Grammar School
Ningyuan Li, Auckland Grammar School
Yixing Li, Auckland Grammar School
Milidu Ratnayake, Auckland Grammar School
Zhifei Shen, Auckland Grammar School
Stephen Zhu, Auckland Grammar School
Kelvin Gong, Burnside High School
Joanna Li, Diocesan School for Girls
Lydia Watson, Diocesan School for Girls
Andrey Borro, Glendowie College
Matthew Fraser, James Hargest College (Senior Campus)
Boen Deng, John McGlashan College
Ricky Lu, Kristin School
Edward Liu, Macleans College
Bryner Lum, Macleans College
Richard Tang, Macleans College
Samuel Chen, Pakuranga College
Claire Shi, Palmerston North Girls' High School
Ju-Eun Kim, Pinehurst School
Vivien Huang, Rangitoto College
Seungjun Bang, St Andrew's College
Stacey Tian, St Cuthbert's College
Bill Yang, St Kentigern College
Alec Van Helsdingen, St Peter's College (Epsom)
Joshua Hogan, Te Aho o Te Kura Pounamu

Year 10 (Form 4) Prize Winners

First	Daniel Jeong	Hillcrest High School
Second Equal	Byung-Hoon Cho	Auckland Grammar School
	Christopher Brown	Christ's College

Top 30 (in School Order):

Maxwell Benson, Auckland Grammar School
Kevin Huang, Auckland Grammar School
Sang Kim, Auckland Grammar School
Miles Lee, Auckland Grammar School
Ajay Shanmuganathan, Auckland Grammar School
Hamish Duncanson, Bethlehem College
Tyla Gartner, Bethlehem College
Jasmine Warner, Howick College
Daniel Davis, Huanui College
Reka Norman, Huanui College
Atrey Gajjar, Lynfield College
Kate Heslop, Lynfield College
Vlad Necsulescu, Macleans College
Sunny Wang, Macleans College
Hannah Williams, Macleans College
Zachary Wong, Macleans College
Ryojin Shiona, Middleton Grange School
Finn Thompson, Nelson College
Sean Lau, Otago Boys' High School
Malachi Hill, Palmerston North Boys' High School
Nico van Wijk, Pinehurst School
Leroy Mangila, Rangitoto College
Cameron Low, St Kentigern College
Kevin Shen, St Kentigern College
Patrick Heavey, St Peter's College (Epsom)
Jungyoon Park, Westlake Girls' High School
Jonathan Everett, Whakatane High School

Year 11 (Form 5) Prize Winners

First	Henry Jiang	Macleans College
Second	Mary Jiang	Macleans College
Third	Aidan Ogilvie	Middleton Grange School

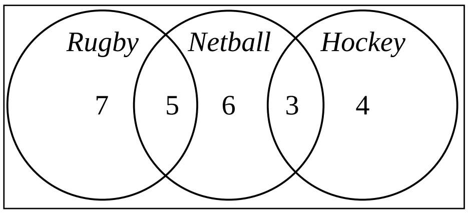
Top 30 (in School Order):

James Brown, Auckland Grammar School
George Easton, Auckland Grammar School
Namankit Gupta, Auckland Grammar School
Marko Ruslim, Auckland Grammar School
Richard Zhou, Auckland Grammar School
Rachel Soohyun Cho, Auckland International College
Simran Shah, Avondale College
Prince Balanay, Botany Downs Secondary College
Jonathan See, Botany Downs Secondary College
Luke Gellen, Christ's College
Brandon Jones, Havelock North High School
Emma Wardle, Hillcrest High School
Anna Redmond, James Hargest College (Senior Campus)
Nic Taylor, John McGlashan College
Victor Chen, King's College
Martin Luk, King's College
Timothy Youn, King's College
Aditya Arora, Macleans College
Saffron Huang, Macleans College
Chris Seong, Macleans College
Annaliese Wheeler, Macleans College
Jemma Zhang, Macleans College
Jonathan Kah, Newlands College
Sheridan Smitham, Palmerston North Boys' High School
Dylan Van Lier, Selwyn College
Daniel Ng, St Kentigern College
Liam McKenzie, St Peter's School (Cambridge)

As usual, in most cases only one solution method is given. It is not necessarily the shortest, nor most elegant.

Question 1 (YEAR 9 AND BELOW ONLY)

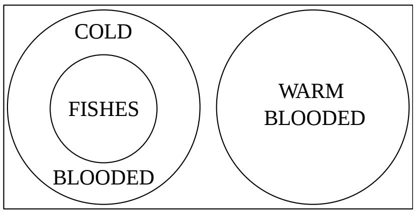
Venn Diagrams are useful for solving certain problems.



(a) The diagram shows a Venn diagram for a class of 30 students and three of the sports they play. For example, the circle labelled 'RUGBY' shows that 12 of the pupils play rugby. Of those 12, 5 also play netball. Nobody in the class plays both rugby and hockey.

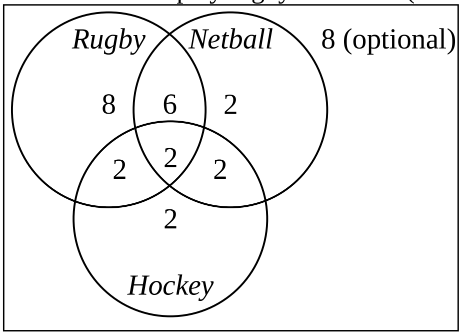
- (i) How many people in the class play netball? **14** *Well answered.*
- (ii) How many people in the class don't play any of these three sports? **5** *Well answered.*

(b) On the answer booklet (not in the questions) draw up your own Venn Diagram with three circles labelled 'COLD BLOODED ANIMALS', 'WARM BLOODED ANIMALS', and 'FISHES' to illustrate the statement 'All fishes are cold blooded animals: some, but not all, cold blooded animals are fishes'. Note: Not all fishes are cold blooded, but for the purpose of the question you may regard them as cold blooded.



Apparently tricky.

(c) On the answer booklet (not in these questions) draw another Venn Diagram for a different class, then answer the question. In this class there are 32 students. 18 play rugby, of whom 8 also play netball and 4 play hockey. 12 play netball, of whom 8 also play rugby (already mentioned) and 4 also play hockey. 8 play hockey, and of those some also play rugby or netball (already mentioned), but 2 students play all three sports.

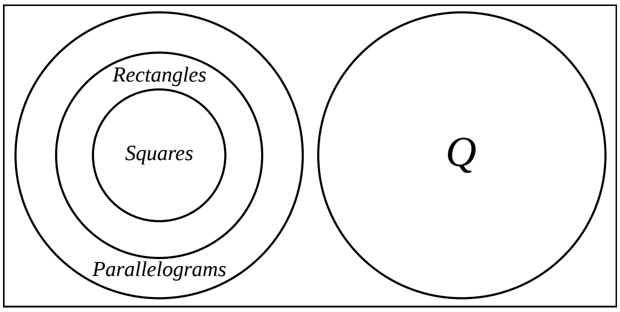


Seemingly tricky. Well answered by some. Many got the diagram correct but not the numbers.

Question: How many students play none of these sports? **8** *Reasonably well answered.*

(d) The following Venn Diagram shows PARALLELOGRAMS, SQUARES, and an unknown quadrilateral called 'Q'. Copy the diagram into your Answer Booklet (do NOT answer on the question sheet), and on <u>your</u> diagram add a circle in the correct place that shows RECTANGLES.

Note: Perfect circles are NOT needed.



Average. Rectangles are 'special' types of parallelograms and squares are 'special' types of rectangles. In some schools almost all students knew this. In others, hardly anyone got it right.

Question 2 (All Years)

The year 2013 has the digits '0', '1', '2', and '3' repeated only once in it.

a) Find a 4 digit number (where the first digit is **not** '0') different to 2013 which uses each of '0', '1', '2', and '3' once and only once. **1023** *is one example of many. Well answered.*

b) Find and list all 4 digit numbers starting with '3' which use '0', '1', '2', and '3' once and only once.
3012, 3021, 3102, 3120, 3201, 3210. *Well answered, although a few students said '6' and didn't list them.*

c) How many 4 digit numbers (where the first digit is **not** '0') are there that use '0', '1', '2', and '3' once and only once? **3 × 6 = 18** *Well answered.*

d) How many 4 digit numbers are there which start with '4', have no repeating digits, and do not use any of '0', '1', '2', and '3' at all? **We can use 5 through 9 for each of the subsequent digits, so 5 choices for first digit, 4 for second, and 3 for third. Thus 60 possibilities in total.** *Fairly well answered. Some listed all such numbers starting with 45 and went from there (so there are 12 such numbers, and a choice of 5, 6, 7, 8, 9 for the second digit makes 60 in total).*

e) How many 4 digit numbers are there altogether that have no repeating digits and do not use any of '0', '1', '2', and '3' at all? **6 choices for the first digit, then 5 for the second digit, 4 for the third digit, then 3 for the fourth digit. So 60 × 6 = 360 possibilities in total.** *Well answered. This is of course 6 times the answer for (d), so follow-on was given.*

Question 3 (All Years)

King Arthur was happiest when his Knights of the Round Table sat around the table in a special way. If there were *n* knights present, he would give them numbers 1, 2, 3, . . . *n*, and then ask them to sit so that each adjacent pair of knights had numbers that summed to the same value as the adjacent pair of knights sitting diagonally opposite them.

The picture in Figure 1 shows the arrangement one day when six knights were present. Here 1 + 6 = 4 + 3, 6 + 2 = 3 + 5, and 2 + 4 = 5 + 1.

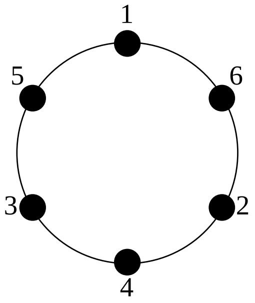
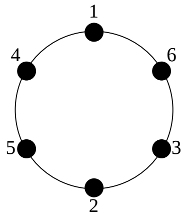


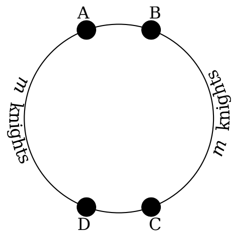
Figure 1

a) Find another arrangement of the six knights that satisfies Arthur's wishes. (Do not use a simple reflection of the one already shown.) You may either draw a circle showing the six numbers, or list the numbers in sequence.



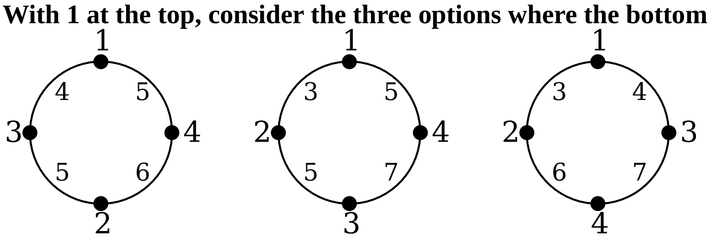
Well answered. The ordered list ***1, 6, 3, 2, 5, 4*** *(or equivalent) also earned full marks.*

b) Why must the number of knights *n* be even for this to work?



If adjacent pair *A* and *B* are *diagonally opposite adjacent pair C and D* as shown, then there must be the same number (say *m*) of other knights sitting between *B* and *C* as there are sitting between *A* and *D*. Then altogether there are 2*m* + 4 knights which is an even number. *The answer 2m + 4 (or equivalent) was not seen often.*

c) Can Arthur's wishes be satisfied with just four knights? (Answer by trying out the possible seating arrangements with 1 at the top. Answers stating 'yes' or 'no' without working will gain no credit.)
With 1 at the top, consider the three options where the bottom is 2, 3, or 4:



(The smaller numbers inside each circle on the diagram shown to the left show the sums of the numbers for each adjacent pair of knights.)

In each case King Arthur’s rule fails to hold. Therefore there is no solution for 4 knights. *Well answered but many missed one case out.*

d) Consider as shown in Figure 2 the three adjacent knights with numbers A, B, and C, and sums s_1 and s_2. (For example, $s_1 = A + B$.) Show that $C - A = s_2 - s_1$.

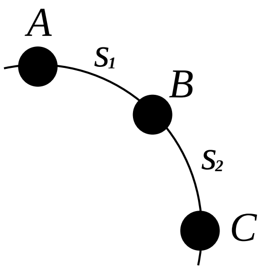


Figure 2

$s_1 = A + B$ and $s_2 = B + C$ so that, subtracting,

$$\begin{aligned} s_2 - s_1 &= (B + C) - (A + B) \\ &= B + C - A - B \\ &= C - A \text{ as required.} \end{aligned}$$

Nowhere did a question give the markers more pleasure. Not well answered by Year 9 students (although some did it correctly) but well answered by Year 11, illustrating the progress students make in generalisation.

e) Suppose four knights are seated as shown in Figure 3. Use the result in part (d) to show that Arthur's wishes would need $A = C$. (This is obviously not possible, hence proving that there is no solution for $n = 4$.)

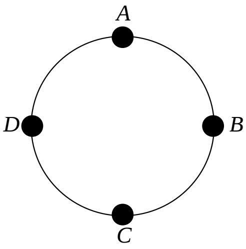


Figure 3

As shown in Figure 4, if $A + B = s_1$ and $B + C = s_2$, then Arthur's rule requires that $A + D = s_2$ and $D + C = s_1$.

Using the result from part d), considering the knights A, B and C, we must have $C - A = s_2 - s_1$, while considering the knights A, D and C, we must have $C - A = s_1 - s_2$. The only way we can have $C - A = -(C - A)$ is if $C - A = 0$, that is $A = C$ (which is impossible).
Not badly done by Year 11.

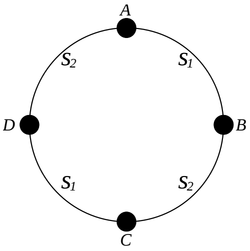


Figure 4

f) Show that there is no solution when n is a multiple of 4.

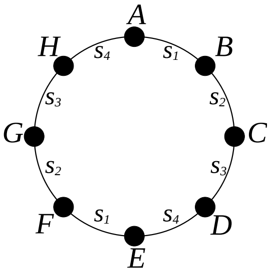


Figure 5

Consider 8 knights as shown in Figure 5, where we are assuming that Arthur's wishes can be met. As before, each number s_n on the table is the sum of the numbers of the adjacent pair of knights; for example, $s_4 = E + D$. Now, looking at the right-hand side of the table,

$$\begin{aligned} E &= s_4 - D \\ &= s_4 - [s_3 - C] \\ &= s_4 - [s_3 - [s_2 - B]] \\ &= s_4 - [s_3 - [s_2 - [s_1 - A]]] \\ &= s_4 - s_3 + s_2 - s_1 + A \end{aligned}$$

So $E - A = s_4 - s_3 + s_2 - s_1$.
However, looking at the left-hand side of the table,

$$\begin{aligned} E &= s_1 - F \\ &= s_1 - [s_2 - G] \\ &= s_1 - [s_2 - [s_3 - H]] \\ &= s_1 - [s_2 - [s_3 - [s_4 - A]]] \\ &= s_1 - s_2 + s_3 - s_4 + A \end{aligned}$$

So $E - A = s_1 - s_2 + s_3 - s_4$. But this is the negative of the value we found above, so, like in part e), we must conclude that $E - A = 0$ and $E = A$ which is not allowed since the knights have different numbers; hence our original assumption is false: it cannot be done with 8 knights.

What makes the arrangement fail with both 4 and 8 knights is that there are an even number of adjacent pairs around each side (left and right) of the table. This makes the difference between the bottom number and the top number (e.g. $E - A$) equal to two sums, each comprising an even number of s_n terms with alternating signs, and with one sum being the negative of the other.

This idea can be easily extended to cover other numbers of knights that are multiples of 4. If n is a multiple of 4 then the number of adjacent pairs around each side (left and right) of the table is $n/2$ which will be *even* — call this k. Then around the right side of the table we have:

$$\textbf{Bottom} = s_k - s_{k-1} + s_{k-2} - \ldots - s_3 + s_2 - s_1 + \textbf{Top}$$

And around the left:

$$\textbf{Bottom} = s_1 - s_2 + s_3 - \ldots - s_{k-2} + s_{k-1} - s_k + \textbf{Top}$$

Giving, as before:

$$\textbf{Bottom} - \textbf{Top} = -(\textbf{Bottom} - \textbf{Top})$$

And hence Bottom = Top which is not permitted.

The most difficult part of this year's competition. Quite a few were able to demonstrate that when there were 8 knights the desired layout was impossible, but the generalisation proved too difficult.

Question 4 (All Years)

A racing track consists of two 100 m straights and two semicircles of 100 m each at each end (see diagram). For the purpose of this question regard it as a line with no width.

a) How many full laps of the track are needed for the 10 000 m race?
10 000 / 400 = 25 *Well answered.*

b) Show that the diameter of the circular sections (i.e. from A to C) must be 64 m, to the nearest metre.
If you cannot work this value out, then use it (if necessary) in the rest of the question. (A, B, C, and D are the corners of the rectangular part of the field inside the track.)
$2\pi r = 200$
$\pi r = 100$
$r = 100 / \pi$
$= 63.694$
= 64 to the nearest m. *Well answered.*

c) Find the total area of the enclosed track. Give your answer in hectares. (A hectare is 100m by 100m).
Area = $64 \times 100 + \pi \times 32 \times 32$
Area = $6400 + 3217$
Area = 9617 m^2 (divide by 10 000)
= 0.9617 ha *Well answered especially by Year 11. Note that this is the answer for when 64m is used as the radius; if 100 / π was used to reach 0.9549 ha full credit was also possible.*

d) How far is it from corner to corner i.e. diagonally from A to D? Give your answer to the nearest metre.
$AD^2 = 100^2 + 64^2$
$AD = \sqrt{14096}$
AD = 119 m (nearest metre) *Well answered.*

Question 5 (All Years)

In this question, give probability answers in decimal form to six decimal places if necessary. For example, if you think an answer is ½, for full marks give the answer as 0.5. (There is no need to give this answer to 6 decimal places.)
A fair chocolate wheel at a school fund-raiser has 20 numbers from 1 to 20. Each number has an equal chance of occurring when the wheel is spun.

a) On one draw, the wheel is spun once. You have bought one ticket; the ticket has one number on it in the range 1 to 20, and that the ticket wins if the wheel stops on that number. What is the probability of your winning?
0.05. *Well answered.*

b) Later the wheel is spun three times. You have one ticket that is valid each time the wheel is spun, i.e. it can win on each and every spin. What is the probability of
(i) zero wins, **$0.95^3 = 0.857375$.**
(ii) exactly one win, **$3 \times 0.95^2 \times 0.05 = 0.135375$**
(iii) exactly three wins? **$0.05^3 = 0.000125$**
Not well answered by Year 9, but by Year 11, many have the 'hang' of probability trees. Part (ii) was often answered without reference to the '3', for partial credit.

c) At the end of the day, the wheel is spun eight times. You have one ticket, valid each time the wheel is spun. On draws one, three, five, and seven, the prize is a large soft toy. On the other four draws, the prize is a small soft toy.
(i) In how many ways can you win exactly one large toy and exactly one small toy?
$4 \times 4 = 16$ Often a list (e.g. WWLLLLLL etc.) was used. *Well answered.*
(ii) What is the probability of your winning exactly one large toy and exactly one small toy?
$16 \times 0.05^2 \times 0.95^6 = 0.029404$ *Difficult but often answered, especially by Year 11.*
Often the '16' was left off, for partial credit.
It may be worthwhile noting that the Year 10 winner answered all of Question 5 correctly (with working) in six lines for a perfect score in the question. Moreover, it wasn't the best we saw!

Hints

1. Show some working. Answers only will not gain full credit, even if correct. One example in the 2013 competition was Question 2(c) where the answer was 18, but if this appeared with no working, some credit was taken off. Even saying 6 × 3 was enough, although normally to be safe you should show a little more in the way of explanation.

   Another example was Question 4(d) where the correct answer was 119 m, found using 'The Theorem of Pythagoras'. Too many people wrote 119 m only and showed no working.

   Some incorrect answers gained partial credit with working, but incorrect answers with no working gained no credit.

   If a question asks for a list of items, you generally (but **not** always) can get away with just the list (especially if it is required for an early part of the question). If the question asks for working giving the correct answer only will generally only earn minimal (and on occasion no) credit.

2. However, do not show too much working. One student wrote four pages for Question 4, and had no time to start any other question.

3. If you get an obviously stupid answer, express some concern. One example was Question 4(a) where you had to divide 10 000 by 400 to get the number of laps. A few people multiplied, and they got 4 million laps. This was clearly stupid. If they had said so, they didn't gain marks, but at least they would impress us with common sense.

   Another example was Question 4(c), the area of a racing track. A couple of students got 1000 million hectares, which is much bigger than Australia. If you get an obviously silly answer, tell us you've made a mistake, and you don't appear silly yourself.

   Yet another example was Question 1(c), where the Year 9s had to state how many people in a class of 32 played no sport. Clearly an answer of 56 made no sense, but the person who reached this answer obviously wasn't thinking about the whole situation.

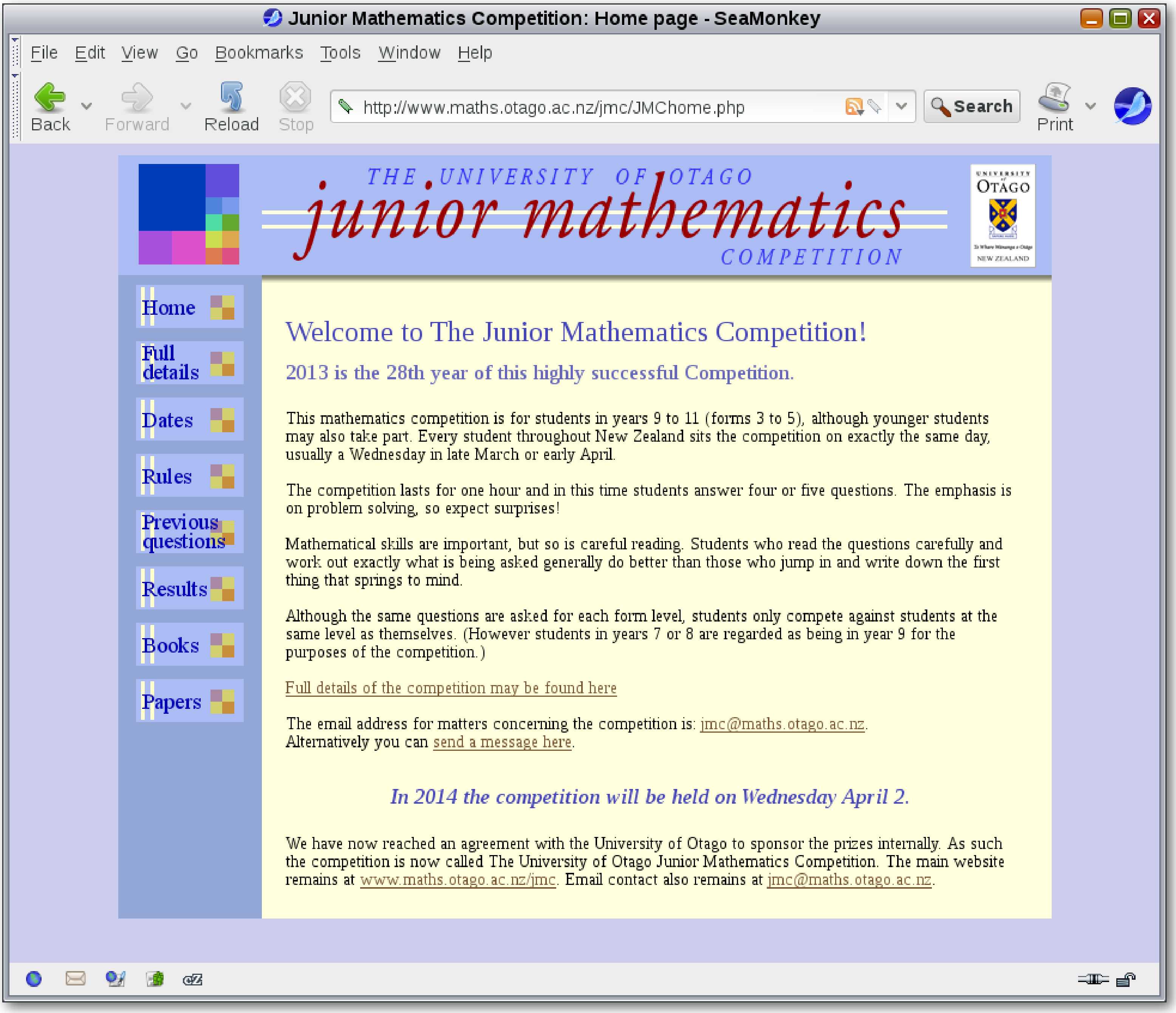
4. Read the question, both before and after you answer it. If we say 'Give your answer to the nearest metre', do it or lose credit. For Question 4(d), the answer was 119 m. Too many students wrote 118.7 m, and lost credit.

5. Don't give answers to 9 decimal places or more. One student had a new calculator and he was proud to give his answers to 15 decimal places! Not sensible. You would need a microscope to measure a running track to 15 decimal places. All he did was lose credit.

6. Do the questions in order. Question 2 was 'easy', but some students started with Question 5, and had no time to do the 'easy' one. You must take care however with this approach (for instance, Question 3 this year was definitely harder than Question 4).

7. Don't answer geometry questions by construction. Any hints that a ruler has been used on a constructed scale diagram will result in 0 marks for the question in general (unless of course the answer has also been reached geometrically). This is a mathematics competition, not an engineering or surveying one!

Don't forget to visit our web page www.maths.otago.ac.nz/jmc



Welcome to The Junior Mathematics Competition!

2013 is the 28th year of this highly successful Competition.

This mathematics competition is for students in years 9 to 11 (forms 3 to 5), although younger students may also take part. Every student throughout New Zealand sits the competition on exactly the same day, usually a Wednesday in late March or early April.

The competition lasts for one hour and in this time students answer four or five questions. The emphasis is on problem solving, so expect surprises!

Mathematical skills are important, but so is careful reading. Students who read the questions carefully and work out exactly what is being asked generally do better than those who jump in and write down the first thing that springs to mind.

Although the same questions are asked for each form level, students only compete against students at the same level as themselves. (However students in years 7 or 8 are regarded as being in year 9 for the purposes of the competition.)

Full details of the competition may be found here

The email address for matters concerning the competition is: jmc@maths.otago.ac.nz.
Alternatively you can send a message here.

In 2014 the competition will be held on Wednesday April 2.

We have now reached an agreement with the University of Otago to sponsor the prizes internally. As such the competition is now called The University of Otago Junior Mathematics Competition. The main website remains at www.maths.otago.ac.nz/jmc. Email contact also remains at jmc@maths.otago.ac.nz.

There you will find, among other things, questions and model answers from recent years.