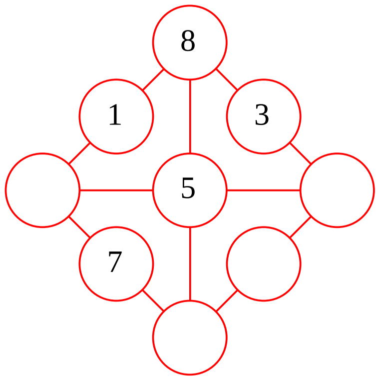
## Solutions and Comments

## Question 1 (Year 9 or below only)

This question turned out to be very easy for most students. Full marks were scored by 60% of the Year 9 candidates. Very few (11 out of 4037) students scored zero.




*The diagram at the right show a Magic Diamond. Each circle is supposed to contain one of the numbers from 1 to 9, with no repeats. However, some numbers are missing from the diagram. In a Magic Diamond the three numbers along each line always add up to the same total.*

*(a)* *Write down in order from smallest to largest the four missing numbers from the Magic Diamond.*

**2 4 6 9** (Half marks were given if the four numbers were correct but in a different order.)

*(b)* *What is the total which the three numbers along each line always add up to?*

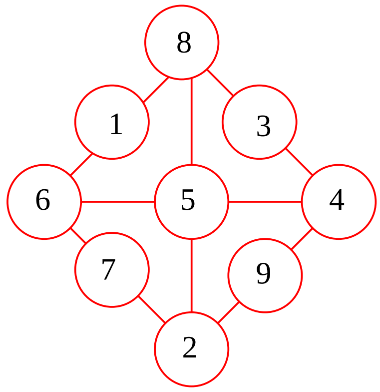
**15**. Reasoning (not required for the competition):

There are six lines each adding up to the same total. If the number 4 is placed in the very bottom circle then the total must be 17. But no other number can now be placed in the very left circle since

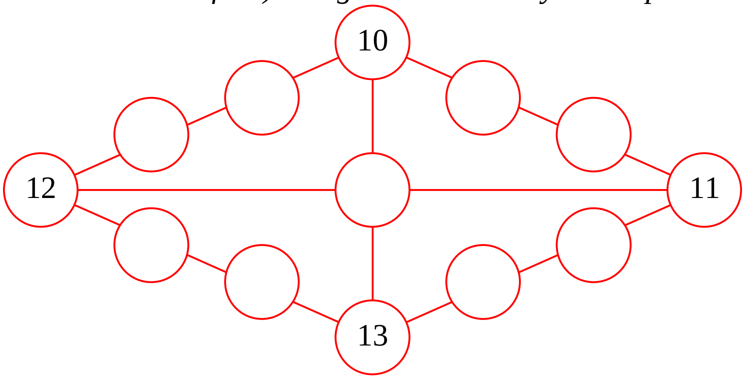
$17 - (8 + 1) = 8$

which has already been used. Similar arguments can be used to lead to the conclusion that 2 is the only number which can go in the very bottom circle, making the total 15.

*(c)* *Carefully copy the Magic Diamond into your Answer Booklet and write in the missing numbers so that the three numbers along each line always add up to the total you wrote down in (b).* (See the answer at the right.)

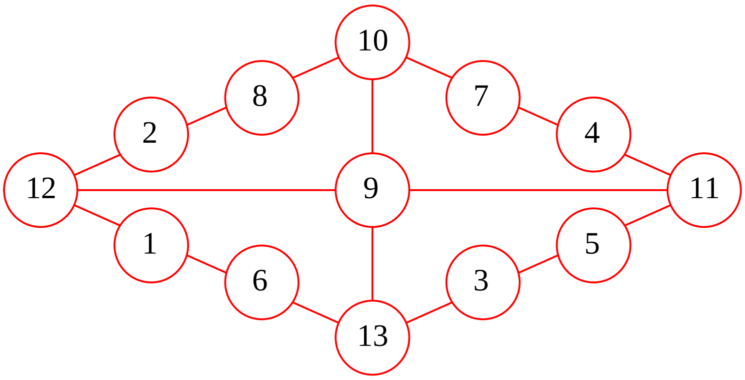



*Marlon tries to draw a larger Magic Diamond. His Magic Diamond contains all the numbers from 1 to 13. Marlon's Magic Diamond is shown below. Once again some of the numbers are missing. The numbers (sometimes three of them and sometimes four) along each line always add up to the same total.*




*(d)* *What is the total which the numbers along each line always add up to?*
**32**. Reasoning (not required for the competition):
Let the very middle number be 9. Then the total along each line must be 32. The sum of the remaining numbers (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) is 36. The four outside lines total $32 \times 4 = 128$. If we subtract the four corners, which each count twice (totalling 92), we are left with 36, so 9 is possible for the middle. Similar arguments show that 8 (or fewer) will not be possible.

*(e)* *Carefully copy Marlon's Magic Diamond into your Answer Booklet and write in the missing numbers so that the numbers along each line always add up to the total you wrote down in (d).*




There are other possible arrangements apart from just swapping pairs of numbers (e.g. 2 and 8), but finding all of these is left as an exercise.

## Question 2 (All Students)

Results in this question were mixed, with many Year 9 students answering (a), (b), and (c) correctly. Yet 7.5% of Year 10 students and 2% of Year 11 students scored zero marks in this question, and 3.5% of all students (mostly from Year 9) did not attempt the question. This was very disappointing since part (a) was designed to be a very easy question testing an essential skill (order of operation). Possibly many students were unable to read the question properly. Overall, only 2.3% of all students scored full marks in this question, and only another 3.5% were able to answer (a), (b) and (c) correctly and then also realise in (d) that a second answer was possible.

*Sally stores mathematical expressions in her calculator. She uses special function buttons* $\boxed{f_1}$ $\boxed{f_2}$ *etc. to store expressions which she wants to use later.*

*One day her little brother Robert is looking at her calculator and he asks Sally about the special function buttons. Sally shows him how they are used (although she doesn't use any fractions at all because Robert doesn't really understand fractions yet). Sally tells Robert that the expression she stored using* $\mathbf{f_1}$ *is* $4x^2$*. This means that if Robert enters a number into this expression, the number will be squared and multiplied by 4.*

(a) *Write down the result when Robert enters the number 3 into the expression Sally stored using* $\boldsymbol{f_1}$*.*

**$4 \times 3^2$ = 36**

This question tested order of operation. Many Years 10 and 11 obtained the correct result, although the incorrect value 144 (found by $(4 \times 3)^2$) was common as well.

(b) *When Robert presses* **INV** $\mathbf{f_1}$ *the calculator undoes the expression stored using* $\boldsymbol{f_1}$*. This means that if Robert uses* **INV** $\mathbf{f_1}$ *for the result he obtained in part (a) then the answer would be 3. Write down the result when Robert uses* **INV** $\mathbf{f_1}$ *for the number 64.*

**The value is 4 because $4 \times 4^2 = 64$**

This question was not answered as well as (a). Common answers were 2 (incorrect order of operation) along with 8 (several students thought that $\sqrt{16}$ = 8), $5.3\dot{3}$ (found by 64 ÷ 12), and 16 384 ($4 \times 64^2$).

*Sally doesn't tell Robert what expression she stored using* $\boldsymbol{f_2}$*, but she does tell him that if he enters 2 and uses* $\boldsymbol{f_2}$ *the result will be 12, while if he enters 3 and uses* $\boldsymbol{f_2}$ *the result will be 36.*

(c) *Find a possible result when the number 4 is entered using the stored result* $\boldsymbol{f_2}$*. Explain what expression has been stored using* $\boldsymbol{f_2}$ *to give your result.*

**Any value from the list in (d) below, along with a correct expression which gives the value.**

There are an infinite number of possible results, which was not realised by many students at all. This means that the list in (d) is not complete. (It is impossible to give a complete list.) Markers were given a preliminary list of values (with expressions) and were asked to add any "new" correct answers which students might come up with. Of course, the expression also has to work for (2, 12) and (3, 36).

In (c), markers gave credit if the expression was expressed in words, e.g. multiply by 24 and (then) subtract 36. However an algebraic expression was required in (d) for full marks. In (c) the markers could also give credit for an iterative answer such as "increase by 24" (12 → 36 → 60) or "mult by 3" (12 → 36 → 108) but this sort of answer was not given full credit in (d).

(d) *Is the result you found in (c) the only possible result? If it is, explain why. If it isn't, write down another possible result when the number 4 is entered using* $\mathbf{f_2}$ *and also give the expression you used to obtain your new result.*

**No, this is not the only result.**

Students had to state this to give themselves the chance to earn full marks. Only a few students (2 out of 830 in one box of scripts) drew the points (2, 12) and (3, 36) on a graph and then stated that you could either draw a straight line through them or an infinite number of curves. Students who said "There may be another value but I can't find it" received credit for, effectively, having an open mind, but the vast majority of students thought that their answer in (c) had to be the only value, and they were given 0 marks for part (d). Rearranging their first expression (e.g. $x^2 + x^3$ for $x^3 + x^2$) received no credit.

The table below gives the original list given to the markers. The most common correct answers (in either (c) or (d)) stated by students were 60, 80, 108, and 144. Nobody gave the expression shown for 48, although many gave 48 as an answer.

| Value | Expression (or equivalent) |
|---|---|
| 48 | $\frac{-144}{x} + 84$ |
| 60 | $24x - 36$ |
| 70 | $5x^2 - x - 6$ |
| 72 | $6x(x - 1)$ or $6x^2 - 6x$ |
| 74 | $7x^2 - 11x + 6$ |
| 76 | $8x^2 - 16x + 12$ |

(Any expression of the form $ax^2 + bx + c$ with $c = t$, $a = 6 + \frac{1}{6}t$, $b = -6 - \frac{5}{6}t$, $t \in \mathcal{R}$.)

| Value | Expression (or equivalent) |
|---|---|
| 80 | $x^3 + x^2$ or $x^2(x + 1)$ |
| 108 | $4 \times 3^{x-1}$ or $\frac{4}{3} \times 3^x$ |
| 132 | $\frac{1}{2} \times 4^x + 4$ |
| 144 | $6x!$ |

There were many other values and expressions given by students. The one that seemed to impress the markers the most was $x^x + x + 6$, which gives the value 266. (About ten students altogether gave this result.) The next table gives some of the other results (in no particular order) reported by the markers. Some of these values are not whole numbers, which strictly speaking was not possible in the question because Robert "doesn't really understand fractions yet", but markers still gave full marks for them:

| Value | Expression (or equivalent) |
|---|---|
| 69.6 | $4.8x^2 - 7.2$ |
| 108 | $24x^2 - 96x + 108$ |
| 78 | $x^3 + 5x^2 - 6$ |
| 260 | $x^{5-x} + x^x$ |
| 73 | $6.5x^2 - 8.5x + 3$ |

(The student who gave this answer said that he had used his graphics calculator, but he did not explain how he did this.)

| Value | Expression (or equivalent) |
|---|---|
| 78.563... | $1.835x^{2.71}$ |
| 53.03... | $-29.0283 + 59.191 \log_e x$ |

(Strictly speaking these last two are only approximate, but the markers gave them full marks.)

| Value | Expression (or equivalent) |
|---|---|
| 96 (this was a common answer) | $\frac{3}{2}x \times 2^x$ or $(12x)2^{x-3}$ |
| 81.6 | $1.2x(x^2 + 1)$ |
| 62 | $x^2 + 19x - 30$ |
| 48 | $(x^3 - x^2)(5 - x)$ |

The list is endless, of course, which indicates how misguided the students were who thought that they had found the only possible answer in (c).

It is obvious from the working shown on the scripts that many students used "guess and check" when they were trying to find their expression. A more reliable approach is to set up a system of simultaneous equations and solve them to obtain your expression. For example, suppose you think a possible expression might have the form $y = ax^2 + b$. Substitute (3, 36) and (2, 12) into this expression to obtain

$$36 = 9a + b$$
$$12 = 4a + b$$

Subtract the second equation away from the first to obtain $a = 4.8$, and you soon reach the first result in the second table above. Of course, the algebraic solving of the simultaneous equation can be done using a graphics or CAS calculator, once the equations have been developed.

After the competition was over, the examiners thought of three more interesting answers. No students gave any of these:

| | |
|---|---|
| No value | $\dfrac{12x}{4 - x}$ |

(This expression cannot work for $x = 4$ because of division by 0, but it gives the correct values for $x = 2$ and $x = 3$.)

| | |
|---|---|
| 12 | $-24x^2 + 48x - 180$ |

(This expression represents a parabola with (3, 36) as the maximum point.)

| | |
|---|---|
| –84 | $-48x^3 + 360x^2 - 864x + 684$ |

(This expression represents a cubic with (2, 12) and (3, 36) as turning points.)

(Teachers might like to consider giving more senior students the following questions as problems, though you should probably try to solve the problems yourself first!

(a) Find the equation of a parabola which has (3, 36) as its turning point and which also passes through (2, 12)

(b) A cubic function has (2, 12) and (3, 36) as turning points. Find the co-ordinates of the point of inflection. (You do not need to find the equation of the cubic.

(c) Find the equation of the cubic which has (2, 12) and (3, 36) as turning points.)

## Question 3 (All Students)

Part (a) of this problem proved to be the hardest question in the whole competition. The main difficulty seemed to be in drawing a useful diagram (problem solving technique: draw a picture) with the earth as a circle (many students drew it as flat) and showing the radius inside the circle (many students did not have *r* on their diagram at all). The question writer had expected that some students would get started but would not be able to handle the algebra involved, but this was not the case at all. The few students who were able to write a correct Pythagorean statement virtually all solved the problem. One or two did get started but failed to finish, but these students seemed to have trouble with the units rather than with the algebra. (One student failed to change 660 m into km and left 660 in his version of equation * shown below. He seems to have given up on the question once he discovered that the value of the radius he was obtaining was not close to 6000 km.)

It is quite possible, of course, that many students do not know what a "sphere" is.

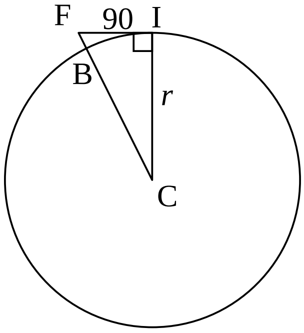
Overall, no Year 9 student scored 100% in this question, and only five Year 10 students (including the Creative Solution Prize winner) and 22 Year 11 students managed to score 100%. At the other extreme 29% of all candidates were given scores of zero by the markers, and 51% did not attempt the question at all.

In the diagram and working below, F stands for the top of Flagstaff, B is the "Base" of Flagstaff at sea level, I is the iceberg, C is the centre of the Earth, and *r* is the radius.

*For this question, assume that the planet Earth is a perfect sphere. During November 2006, residents of Dunedin were able to take helicopter rides to look at several icebergs which were floating off the coast.*

*(a) One day, when conditions were clear and sharp, viewers were able to see an iceberg on the horizon from the top of a local hill, Flagstaff, which is 660 m above sea level. If the distance in a direct line from Flagstaff to the iceberg was estimated to be 90 km, use this information to show that the radius of the Earth from the centre of the planet to sea level is between 6000 km and 7000 km.*

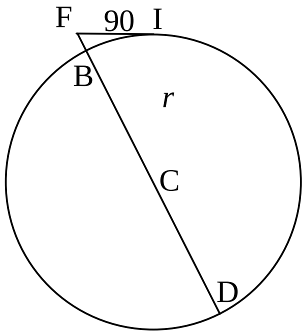
The diagram shows the iceberg on the horizon (surface of the Earth) with the top of Flagstaff 0.66 km above the Earth. The diagram is clearly not to scale. Once the diagram is drawn, students needed to realise that FI is a tangent to the circle, and so ∠FIC = 90° (**tangent is perpendicular to radius)**. This needed to be stated for full marks. A few students wrote their Pythagoras statement without this justification and still scored good marks in the question.




$$(r + 0.66)^2 = r^2 + 90^2 \qquad *$$
$$r^2 + 1.32r + 0.4356 = r^2 + 8100$$
$$1.32r = 8099.5644$$
$$r = 6136 \text{ km}$$ (nearest km)

(Note that this is not the real (average) radius which is given as 6370 km by the Eton Statistical & Maths Tables, but it is as accurate as possible from the given information. The values given in the question (660 m and 90 km) were those suggested by the media at the time, so we decided to stick with these values.)

Of course it is possible to solve the equation * using a CAS calculator. This was not observed by any of the markers. The few students who wrote out equation * all seemed to be able to solve it with full working shown, apparently without using technological help.




One Year 10 student found a completely different solution method and he has been awarded a special prize, the "Creative Solution Prize" mentioned above. His method depends upon knowing a property concerning the chord of a circle intersecting with the tangent. He extended the line BC to the opposite side of the circle so that he now had the chord BD (which has length of 2$r$). For this method it is not necessary to establish that ∠FIC = 90°.

This method gives $\quad FB \times FD = FI^2$

$$0.66(0.66 + 2r) = 90^2$$

Solving this equation gives $r = 6136$ km.

It is also possible to test both 6000 and 7000 (the values given in the question) in equation *. To do this you substitute 6000 into the left hand side (LHS) of * to obtain 36 007 920.44. When you substitute 6000 into the RHS of * the result is 36 008 100. So using 6000, the LHS of * is **smaller** than the RHS.

Now substitute 7000 into *. The two results are LHS = 49 009 240.44 and RHS = 49 008 100. This time the LHS result is **larger** than the RHS result.

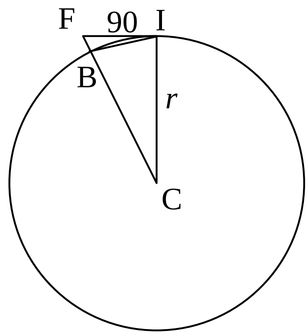
Since the LHS for 6000 is smaller than the RHS, but for 7000 the result is "the other way around", we are able to conclude that somewhere between 6000 and 7000 there **must** be a (real) value of $r$ which makes the LHS **equal** to the RHS. This approach was tried by a few students, one or two of whom convinced the markers they knew what they were talking about. Of course, to succeed here you first need to have a correct equation such as equation * set up, and you may also need a calculator. (This method does not actually calculate the value of the radius, but students were able to use the given value 6000 in part (b), meaning that it was possible in theory to earn full marks in Question 3 without calculating the radius at all.)

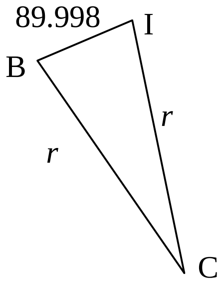
The organisers of the competition acknowledge that a calculator might give an advantage to students in this particular question. One or two students, particularly Year 9 students, wrote on their papers that they did not have a calculator and so they couldn't answer this question. (One student wrote this but still managed to obtain a MERIT award for her work in other questions.) However the only point where a calculator might be needed is when the equation $1.32r = 8099.5644$ (or $1.32r = 8100$ if the student realises that the value $0.66^2$ is negligible compared to the other values in the question) has to be solved. Even this could be solved by testing the values 6000 and 7000 and showing that the solution must lie between these extremes. This would require multiplication of 6000 by 1.32, and based on the evidence in (b) below, where $2 \times 6000 \div 12$ proved to be difficult for some students, the long multiplication of $6000 \times 1.32$ without a calculator might be too hard for many.

With a potentially wide range of calculators available, from CAS through to no calculator at all, the competition organisers are sometimes finding it difficult to write calculator neutral questions. It would have been sad to abandon the interesting and topical iceberg question simply because some students do not possess a calculator. It was hoped that the careful wording of part (b) in particular should have enabled students without calculators to make some progress in the question. In the end a reasonable percentage of students (about 20% altogether) did score some marks in the question (mostly in (b)) but we cannot tell what percentage did score marks but did not have a calculator.

"Solution methods" were seen which sometimes led to a seemingly correct answer but which were only given a small amount of credit by the markers. These "methods" are either based upon an incorrect assumption about right angled triangles or upon seeing similar triangles where none exist.

Some students assumed (without justification) that $\angle$FBI = 90°. They then used the Theorem of Pythagoras to show that BI = 89.998 (3 d.p.). Now they have an isosceles triangle BIC which has the radius of the Earth as two of its sides.



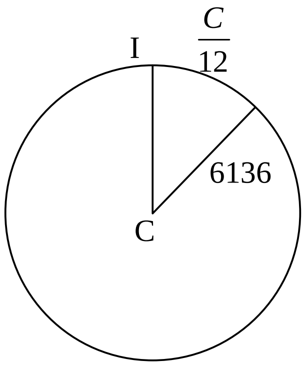

The trouble with this triangle is that $\angle$IBC must be 90° (angles on a straight line) and so the two equal sides (BC and IC) in the "isosceles triangle" can never meet. Therefore the original assumption (that $\angle$FBI = 90°) must be incorrect. This fact did not prevent students from pressing on, and some of them did eventually (somehow) obtain a radius value very close to 6136 km. Some (small) credit for the answer may have been given by markers, but not for the method.

The other approach seen was to claim that there are similar triangles (usually FIB and BIC) somewhere in the diagram. An equation was then set up based on the ratios of the sides. More than one student wrote down the equation $\frac{90}{0.66} = \frac{r}{90}$ based on "similar" triangles. The solution to this equation is $r = 12\,272.72$. The students then seem to have decided that there must be an arithmetic error because the question said that $6000 < r < 7000$. So they divided by 2 to obtain 6136.36 which is remarkably close to the correct value but which has at least two errors leading to the result. Who said that marking mathematics papers is easy?

Teachers who use this question in class might like to follow it up by asking how far away a boat is if it can be seen on the horizon by someone standing on a beach with their eyes at a height of 2 m. The answer is about 5 km using the true radius of the Earth (3370 km). This was the original version of the question until the icebergs were seen off the Dunedin coast, at which point the question was extensively rewritten.

(b) This part of the question was answered correctly by many students, although several used the incorrect formula $A = \pi r^2$ (zero marks) rather than $C = 2\pi r$ for the circumference. Most students used the values given in the question because they had not been able to calculate $r = 6136$ km in (a).

*If the iceberg broke off from Antarctica and then travelled in a straight line (relative to the surface) covering one twelfth* $\left(\frac{1}{12}\right)$ *of the circumference of the Earth before it was seen near Dunedin, find the distance it had* *travelled before it was seen. (Note that if you do not have a calculator then you may use the values 6000 km for the radius of the Earth and $\pi$ = 3.14.)*




$$C = 2\pi r$$
$$= 2 \times \pi \times 6136 = 38\ 554$$
$$\therefore \text{ distance } = \frac{38\ 554}{12} = 3213 \text{ km}$$

Those students who used the values $r$ = 6000 and $\pi$ = 3.14 were able to gain full marks for calculating $(2 \times 3.14 \times 6000) / 12 = 3140$ km.

A calculator should not be needed for this (after all, what is $2 \times 6 \div 12$?), although from what we could see, most students seemed to use one, although some students without one did not notice how simple the arithmetic really is. One example can be seen in the work of one Year 9 student. He does not appear to have had a calculator and because of this he made an error which cost him a few marks, but not ultimately a place in the Top 30. He wrote:

$2 \times 6000$
$= 12000$
$12000 \times 3.14$
$= 37480$ (There is an error here.)

He now carried out some long division correctly:

$12\overline{)37480}$ giving $3124$ and he had a remainder of 4.

His final result was the incorrect value $3124\frac{1}{3}$. Had he realised that the result of all his work should have been simple multiplication by 1000 he would not have made the error which cost him a couple of marks.

The very common answer 3141.592654 km does not make sense as it suggests accuracy to the nearest tenth of a cm.

One student complained that he was unable to answer the question because it required the value from (a) which he had been unable to obtain. This is clearly incorrect because the value 6000 was suggested to cover for this eventuality. Perhaps he didn't think the hint applied to him since he had a calculator. (This comment reveals a poor assessment technique. Students should learn that often assessment questions are linked by context, but may be independent as far as working is concerned. On the other hand, early questions can sometimes be used to suggest possible approaches for solving later problems. This is certainly the case in Questions 4 and 5 below, where part (a) of each question should be helpful later on.)

Another student thought that the iceberg travelled 0.8 km before being seen. This means that Dunedin must be very close to Antarctica, or perhaps the student just thinks that Dunedin is cold.

The question had been simplified from the original situation. It is believed that the real icebergs may have broken off the Antarctic coastline on the other side of the planet, but for the competition the question was simplified (straight line) rather than following a curved path halfway around the world.

## Question 4 (All Students)

This question gave many students the chance to earn credit after they discovered that they were having trouble with Question 3. Parts (a) and (c) were very easy for anyone who realised that each game eliminated one player. This means that with 35 players there would be one winner and 34 eliminated players. Therefore 34 games are needed. This is an example of using a simpler case to solve a more difficult problem because in (c) the same logic means there will be 2006 games needed. Most students did not realise this and full page answers with very large tree diagrams (and incorrect results) were common.

Overall, about 27% of students scored zero for this question, 54% scored some marks, just over 1% scored full marks, and 18% did not attempt the question.

*The 2007 Kakanui Bowls tournament was a knock-out competition. This means that if there were exactly 32 competitors then the first round of games consisted of each person playing against someone else, with the loser of each game being eliminated. The 16 competitors left then played in the next round, with the loser of each of these eight games being eliminated. This arrangement continued until only two players were left. The winner of this final game won the tournament.*

*However if there were extra competitors, for example an extra three (making 35 altogether), then six competitors first of all had to play a preliminary game (three games would be needed) to select the three competitors who would go through to be part of the last 32.*

(a) *How many games were there altogether if there were 35 competitors?*

**34**.

For those who did not see the quick method described above it is possible to work the answer out using a longer method. Firstly there are three preliminary games (not 6) to reduce the number of competitors to 32, which is a power of 2 (since $2^5 = 32$). The next round will eliminate 16 players, then 8, and so on. So the required number of games is:

$$3 + 16 + 8 + 4 + 2 + 1 = 34$$

(b) *When the entries closed it was discovered that there were 2007 competitors. The knock-out method described above was employed for this larger number of competitors. Boris, the eventual winner of the tournament, was one of those who first had to play a preliminary game. How many games did Boris have to play?*

**11.**

For this part of the question recognising powers of 2 was useful.

$2^{10} = 1024$ and $2^{11} = 2048$ so Boris played one game to reach the last 1024 players, then 10 more games.

(c) *How many games were there altogether for this number (2007) of competitors?*

**2006.** (See above for details on how to solve this problem.)

(d) *If there are n full rounds of games plus m preliminary games, write down an expression for the total number of games which must be played in the tournament.*

$\mathbf{2^n - 1 + m}$

This part of the question was poorly answered and is the main reason why only 1% of students scored full marks in Question 4. Not many students realised that the number of games being played involved a power of 2, and answers like $n + m$ were common (for zero marks). It was also obvious that many students failed to realise that a general answer was needed, not an answer relating to 2007. Answers like $2006n + m$ earned no credit.

An alternative answer was possible for full marks, and several students gave this:

$$m + 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^2 + 2^1 + 2^0$$

**Question 5 (All Students)**

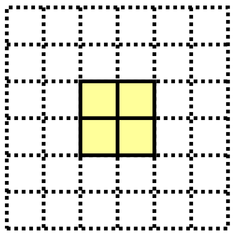
This question was too difficult for most candidates, although not as hard as 3(a). Markers were instructed to give the benefit of the doubt in (a) if they thought the students were on the right track. However, in (b) and (c) the answers had to be very good indeed to gain full credit. Many students drew diagrams but did not write any explanation and this usually earned zero marks. On the other hand, one or two students wrote beautiful answers without drawing a diagram at all.

One student wrote that the wording of the question was too difficult to understand. But it is the final question in the competition and the competition checkers were all happy with the wording. (The question itself is based on an international problem solving competition so it will have been extensively checked there as well.) The student concerned then ruined the effect of his statement by claiming that all those in his immediate vicinity also thought that the question was too hard to understand. This raises some concern about the integrity of the supervision at his particular school. Despite this concern the competition organisers decided not to withdraw the student's MERIT award. After all, he was the only student from his school to gain an award so he can't have copied off anyone else, and since he was the only award winner from that school all the others must have had trouble understanding the other questions as well. Hopefully supervision of the competition will be stricter at that school in the future.

As with Question 4, a "simpler case" is introduced in part (a) to help students deal with the more complicated situation in (b).

*(a)* *Paul has a sheet of graph paper which has 6 squares along each side. He draws squares of side lengths 1 through to 5 on this paper, along the lines. Show that no matter how he does this there will be a square on the paper which is part of at least three of the squares that he has drawn.*

To be certain of gaining full marks here, students needed to realise how important the "central" four squares of the six by six square are (see the diagram). No matter where we place the $5 \times 5$ and $4 \times 4$ squares, they **must** cover all of this central two by two block.



Now, no matter where the $3 \times 3$ square is placed, it must cover (at least) one of the four central squares. So at least one of the unit squares in this central two by two block must always be covered by at least three of the squares. (The $2 \times 2$ and $1 \times 1$ squares do not matter as they may be placed anywhere.)

Note that the common "Pigeon Hole" approach on its own does not earn any credit here. This argument is based on numbers only. The idea is that the $6 \times 6$ square has area 36, and the sum of the other squares is 55 (alternatively the largest two squares sum to 41). Therefore at least 19 (or 5) squares must be covered by more than one square. But on its own this doesn't establish that **three** squares are covered unless further arguments are developed. The Pigeon Hole approach ignores the **packing** of the squares.

*(b)* *Paul now takes another sheet of graph paper which has 40 squares along each side. On this sheet he draws squares of side lengths 1 through to 39 on this paper, along the lines. Show that no matter how he does this there will be a square on the paper which is part of at least 20 of the squares that he has drawn.*

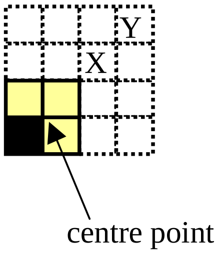
A similar argument to that in (a) applies. The central two by two block must be completely covered by each of the $39 \times 39$, $38 \times 38$, ... , $21 \times 21$ squares no matter where they are placed. So all of the central two by two block is completely covered by 19 squares. Now, no matter where the next $(20 \times 20)$ square is placed, it **must** cover (at least) one of the four central squares. So at least one of the unit squares in this central two by two block must be covered by at least 20 of the squares. (The smaller squares, $19 \times 19$, $18 \times 18$, ... , $1 \times 1$, do not matter as they may be placed anywhere, and this may or may not increase the number of squares covered by at least 20.)

It is worth noting that a few students wrote quite vague answers in (a) but as they were thinking about they problem they obviously sorted out the issues and then wrote excellent answers in (b). Giving the simpler case scenario in (a) clearly helped these students.

*(c)* *For the situation in (b), is it possible for there to be exactly one square of this sort, i.e. exactly one square which is part of at least 20 of the squares that he has drawn?*

Students first needed to be able to explain exactly how to place the largest squares so that exactly one square is covered by 20 squares, then explain how to avoid covering any more squares so that no more squares are covered by 20.

If all the squares from $39 \times 39$ down to $20 \times 20$ are placed touching one of the corners (e.g. the bottom left corner), the bottom left square (shaded in the diagram) in the central two by two block will be covered by exactly 20 squares, and no other square will be covered by 20 squares.




This means we now have to place the remaining squares ($19 \times 19$ down to $1 \times 1$) in such a way that they do not cover the square already covered by 20 squares, nor cover any other square so that a second square becomes covered by 20.

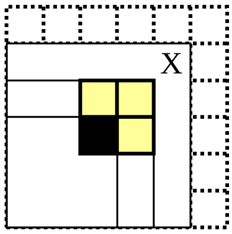
This can be done. One way to do this successfully is to start from the opposite (top right) corner. From here, the $19 \times 19$ square will not cover any part of the two by two central block. This means that the square marked X will be covered by 19 squares, not 20. Now the $18 \times 18$ square is placed touching the top right corner, and the square marked Y becomes covered by 19 squares.

Continuing in this way results in no more squares being covered by 20 apart from the shaded one.

Markers reported seeing one or two cases of students who explained part (c) perfectly, but had scored nothing in (a) or (b). Perhaps part (c) is easier for students to explain than parts (a) and (b). After all, parts (a) and (b) require generalisation. The words ‘no matter where the squares are placed” were a good sign in (a) and (b) that the student was not restricting their argument to a particular arrangement of squares. But in (c) the student only had to describe one way to place the squares, which may be an easier skill.

Another possibility is that by the time students had reached part (c), they had thought through the situation and had a better grasp of the problem.

A similar argument based on the simpler 6 squares case could earn half marks. After the $5 \times 5$, $4 \times 4$, and $3 \times 3$ squares are placed touching the bottom left corner, one square (shaded in the diagram) is covered by exactly three. Now place the $2 \times 2$ square touching the opposite (top right) corner. One square (marked X) now has two squares covering it, but not three.




Placing the $1 \times 1$ square in the top right corner (or elsewhere in the strips at the top or the right) means another square has two covering it, but there is still only the one (shaded) square covered by exactly three.

The question setter had hoped that students might use the simpler 6 by 6 case in part (c) to lead them into the more complicated 20 by 20 case. However this was not observed, possibly because most students had run out of time when they tried the very last question in the competition.

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