Department of Mathematics and Statistics

UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

Junior Mathematics Competition 2023

Questions for Part 2

Instructions to Candidates

You have a maximum of **fifty minutes** to answer **six** questions out of **eight**. The set of questions you answer is determined by your year level:

Question 1: 10 marks. Year 9 and below only.
Question 2: 10 marks. Year 10 and below only.
Question 3 to Question 6: 20 marks each. All students.
Question 7: 10 marks. Years 10 and 11 only.
Question 8: 10 marks. Year 11 only.

If you answer an incorrect question for your year level it will not be marked.

These questions are designed to test your ability to analyse a problem and express a solution clearly and accurately.

Please read the following Instructions carefully before you begin.

1. Do as much as you can. You are not expected to complete the entire paper. In the past full answers to three full (20 mark) questions have represented an excellent effort.
2. You must explain your reasoning as clearly as possible with a careful statement of the main points in the argument or the main steps in the calculation. Generally even a correct answer without any explanation will not receive more than half credit. Likewise clear and complete solutions to three full problems will generally gain more credit than sketchy work on four.
3. Credit will be given for partial solutions and evidence of a serious attempt to tackle a problem.
4. Textbooks are NOT allowed. Calculators may be used and students who do not have one may be disadvantaged. Otherwise normal examination conditions apply.
5. We recommend black or blue pens. Dark pencil is acceptable if you have nothing else. Do NOT use red or green pens, or light pencil that we cannot read.
6. We will penalise inappropriate rounding and incorrect or absent units.

Note: There are **four** pages in this question booklet: this instruction page and **three** pages of questions.

DEFINITION

A *prime number* has exactly two factors, 1 and itself. By this definition, 1 is not a prime number.

The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.

DO NOT TURN OVER UNTIL TOLD TO DO SO.

Question 1: 10 marks (Years 9 and below only)

Farah has an appointment across town from her office. Normally she could drive to it by just going North a certain amount then West a certain amount, but today the town is full of traffic works and many of the roads are blocked off.

Fortunately her car's GPS can navigate her through the blocked road system. From her office, it tells her to take the route described in the box to the right.

(1) From your office, drive 51 metres North.

(2) Drive 75 metres East.

(3) Drive 59 metres North.

(4) Drive 120 metres West.

(5) Drive 45 metres South.

(6) Drive 111 metres West to your destination.

(a) How many metres does Farah drive in her car in total?

(b) If none of the roads were blocked and Farah could take her normal route (so North a certain amount then West a certain amount), how far in metres would she have had to have driven to get to her appointment?

(c) Suppose there was a road that went Northwest from Farah's office to where her appointment was. How far in metres would she have had to have driven between her office and where her appointment was if she took this road?

Question 2: 10 marks (Years 10 and below only)

Using the constraints listed below, find values for the variables in the following 3 by 3 table:

a	3	c
4	e	f
g	h	i

(a) Each of 1, 2, 3, 4, 5, 6, 7, 8, and 9 appear exactly once in the table.

(b) e is even while h is odd.

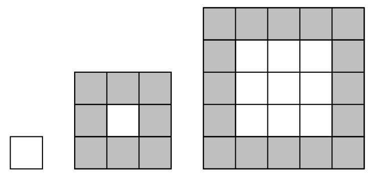
(c) $a < c$ and $g > i$.

(d) There is no row in the table with an odd number of even numbers.

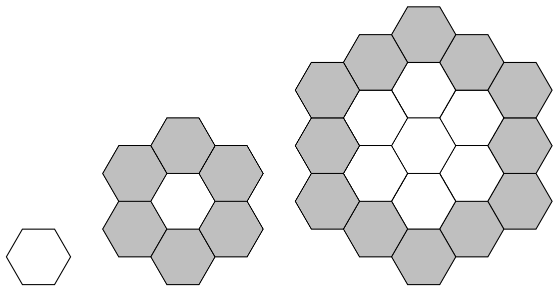
(e) The sum of the first row is 9, while the sum of the last row is 19.

Question 3: 20 marks (All Years)

Hahona builds up a polygon by taking a square and surrounding it by other squares of the same width, height, and orientation such that each corner or edge of the original square is adjacent to a corner or edge of another square. He then repeats the process, at each step creating a larger polygon. Hahona also does the same thing with regular hexagons — see the figures to the right.



$n = 1$ $n = 2$ $n = 3$



$n = 1$ $n = 2$ $n = 3$

In the below table n represents a stage of the polygon construction (with $n = 1$ representing the base square or regular hexagon), s represents the number of squares used in this stage, and h represents the number of hexagons used in this stage.

n	1	2	3	4	5
s	1	9		49	
h	1	7	19		

(a) Find s for (i) $n = 3$ (ii) $n = 5$

(b) Find h for (i) $n = 4$ (ii) $n = 5$

(c) Find the smallest n such that (i) $s > 300$ (ii) $h > 200$

(d) If the perimeter of the regular hexagon is 6 cm, find the perimeter of the polygon constructed using the regular hexagon when $n = 5$.

Question 4: 20 marks (All Years)

In this question n is an integer greater than 0, and p is a prime number.

For a given n let p be the nth prime number and $s(n)$ be the distance from p to the square number closest to p. For example, if $n = 4$ then $p = 7$. The closest square numbers to 7 are 4 and 9, and since 9 is closer to 7 than 4 we have $s(4) = 9 - 7 = 2$.

(a) For the following values of n and p find v (the largest square number smaller than p), w (the smallest square number larger than p), and $s(n)$:

  (i) $n = 7, p = 17$
  (ii) $n = 10, p = 29$

(b) Find the smallest value of n such that $s(n) > 5$.

A function similar to $s(n)$ is $c(n)$, which for a given n is the distance from p to the cube number closest to p (where again p is the nth prime number). For example, the closest cube numbers to $p = 7$ are 1 and 8, and since 8 is closer than 1 $c(4) = 8 - 7 = 1$.

(c) For the following values of n and p find y (the largest cube number smaller than p), z (the smallest cube number larger than p), and $c(n)$:

  (i) $n = 7, p = 17$
  (ii) $n = 16, p = 53$

(d) Find the smallest value of n such that $c(n) > 10$.

(e) For $n = 1$, $p = 2$ and $s(1) = c(1)$. Can you find another value for n such that $s(n) = c(n)$? If so, list n, p (the nth prime number), $s(n)$, and $c(n)$. If not, briefly explain why no such n exists.

Question 5: 20 marks (All Years)

For a set of **positive real** numbers $x_1, x_2, \ldots, x_n$ we define three functions A, G, and H:

$$A(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n} \qquad G(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \ldots x_n} \qquad H(x_1, x_2, \ldots, x_n) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

The function A refers to the *arithmetic* mean of $x_1, x_2, \ldots, x_n$, while G and H refer to the *geometric* and *harmonic* means of our set respectively. For example, for the set $\{1,2,3,4\}$, $n = 4$, and we have $A(1,2,3,4) = 2.5$, $G(1,2,3,4) = \sqrt[4]{24} = 2.21$ (to two decimal places), and $H(1,2,3,4) = 1.92$. (When we take the nth root for G we ignore the negative root if n is even.)

(a) Find $A(4, 6.3, 7.2)$, $G(4, 6.3, 7.2)$, and $H(4, 6.3, 7.2)$, rounding to two decimal places if necessary. (Here $n = 3$.)

For the set $\{2,4,6,8\}$, $A(2,4,6,8) = 5$, $G(2,4,6,8) = 4.42$ (to two decimal places), and $H(2,4,6,8) = 3.84$. These are twice the values of $A(1,2,3,4)$, $G(1,2,3,4)$, and $H(1,2,3,4)$ respectively.

(b) For any **positive real** number b, show that:

  (i) $A(bx_1, bx_2, \ldots, bx_n) = bA(x_1, x_2, \ldots, x_n)$
  (ii) $G(bx_1, bx_2, \ldots, bx_n) = bG(x_1, x_2, \ldots, x_n)$
  (iii) $H(bx_1, bx_2, \ldots, bx_n) = bH(x_1, x_2, \ldots, x_n)$

(c) Is it possible to find a set of 10 numbers such that their arithmetic, geometric, and harmonic means are all equal? If so, list such a set. If not, briefly explain why no such set exists.

Question 6: 20 marks (All Years)

In this question, we only get out of a lift if we are going into another lift or we have reached our destination.

JMC Towers operates two lifts. Each lift follows a set of rules as to which floors it will stop at:

> *(1)* When going up, lift A can only stop at every fourth floor from where it started, but when going down it can only stop at every fifth floor.
>
> *(2)* When going up, lift B can only stop at every seventh floor from where it started, but when going down it can only stop at every ninth floor.

Suppose that JMC Towers has 30 storeys, with the ground floor being numbered 0 and the highest floor being numbered 29, and assume that neither lift provides access to the basement.

TURN OVER

(a) If you get into lift A on the ground floor, what is the highest floor you can reach if you only use this lift (going up only)?

(b) If you get into lift B on the 29th floor, what is the lowest floor you can reach if you only use this lift (going down only)?

If you wanted to go from the ground floor to the 11th floor without ever going down, one way to do this (which we call a *path*) is to get into lift A going up, get out at the 4th floor when it stops for the first time, then get into lift B going up, and get out when it stops the first time. A shorthand way of stating this is AU1, BU1. To get to the 15th floor from the ground floor (only going up), there are now two distinct paths: AU2, BU1 or AU1, BU1, AU1; we say the first path has two stops, and the second three stops. If we went down from the 14th floor to the ground floor, one path would be AD1, BD1.

(c) Find a path to get to the 29th floor from the ground floor if you can *optionally* change lifts every time a lift reaches a floor, and can only go up.

(d) Is the path you found in (c) the only path you could take with the same conditions applying? If so, briefly explain why this is the case. Otherwise, list another path with the same conditions as in (c).

(e) There exists a path to the ground floor from the 29th floor that takes seven stops, providing that you *must* change lifts every time a lift reaches a floor it can stop at, and you can either go up or down every time you change lifts. Find such a path.

Question 7: 10 marks (Years 10 and 11 only)

Sierpiński's triangle is a variation of Pascal's triangle: we take each number in the latter structure, and if said number is odd, the corresponding number in the former structure is 1; otherwise it is 0. As such, since the first 5 lines of Pascal's triangle are {1}, {1, 1}, {1, 2, 1}, {1, 3, 3, 1}, and {1, 4, 6, 4, 1}, the first 5 lines of Sierpiński's triangle are {1}, {1, 1}, {1, 0, 1}, {1, 1, 1, 1}, and {1, 0, 0, 0, 1}, respectively.

(a) List the numbers in the 6th line of Sierpiński's triangle.

If we take a line of numbers in Sierpiński's triangle, we can use each number as a digit in a binary number, which can then be converted to a decimal number. For example, the 5th line in Sierpiński's triangle reads 1, 0, 0, 0, and 1, which can be represented as the binary number 10001, which in decimal is 17 (since $17 = 1\times 2^4 + 0\times 2^3 + 0\times 2^2 + 0\times 2^1 + 1\times 2^0$).

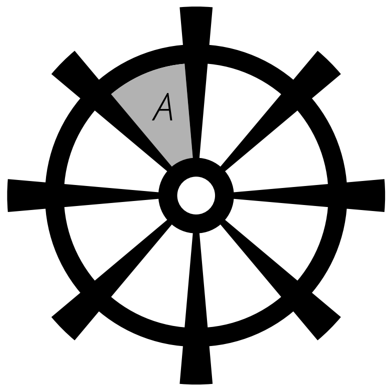
(b) Find the binary representation of the 7th line of Sierpiński's triangle, then convert it into a decimal number. (List both your binary and decimal numbers.)

(c) Briefly explain why the decimal representation of every line of Sierpiński's triangle is odd.

(d) The decimal representation of the 5th line is a prime number. What is the next line of Sierpiński's triangle with a prime decimal representation? (List the line number, the binary representation, and its decimal form.)

Question 8: 10 marks (Year 11 only)

For a school art project Kurt is designing a poster with a nautical theme. On the poster he will draw an abstract representation of a sailing ship's wheel, which can be seen to the right.



The wheel consists of two rings: an inner ring with an inner radius of 2 cm and an outer radius of 4 cm, and an outer ring with an inner radius of 14 cm and an outer radius of 16 cm. There are also 8 spokes, each of which is constructed using a 10° sector of a circle with radius 20 cm, placed so that the sharp end of the spoke is at the middle of the wheel, and then cut off at the outer radius of the inner ring.

(a) To 2 decimal places find the area of the outer ring, including the areas that intersect with the spokes.

(b) To 2 decimal places find the area of one of the wheel's spokes, including the area that intersects with the outer ring.

(c) To 2 decimal places find the area of the empty section of wheel labelled *A* and shaded grey on the diagram.

(d) To 2 decimal places find the area of the ship's wheel (the total area of every part of the diagram shaded black).

(END OF COMPETITION)