Introduction to stochastic partial differential equations

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September 11, 2008

Abstract

We introduce the Hilbert space-valued Wiener process and the corresponding stochastic integral of Itô type. This is then used together with semigroup theory to obtain existence and uniqueness of weak solutions of linear and semilinear stochastic evolution problems in Hilbert space. Finally, this abstract theory is applied to the linear heat and wave equations driven by additive noise.

Contents

1 Functional analysis essentials

In this section we discuss a few concepts and results form the theory of operators in Hilbert spaces. We either give a proof or give a reference to the proof. Consider two separable Hilbert spaces $(U, \langle\cdot,\cdot\rangle_U), (H, \langle\cdot,\cdot\rangle_H)$ where the respective subscripts will be suppressed whenever it is clear from the context which one is meant.

1.1 Spaces of linear operators

Let $L(U,H)$ denote the Banach space of bounded linear operators from U to H. If $U = H$, then we simply write $L(U)$. For $T \in L(U)$ we write $T \geq 0$ if T is self-adjoint positive semidefinite, that is, $T^* = T$ and $\langle Tu,u\rangle_U \geq 0$ for all $u \in U$. Let $L_1(U,H)$ denote the set of *nuclear operators* from U to H, that is, $T \in L_1(U,H)$ if $T \in L(U,H)$ and there are sequences $\{a_j\}_{j\in\mathbf{N}} \subset H, \{b_j\}_{j\in\mathbf{N}} \subset U$ with $\sum_{j=1}^{\infty} \|a_j\|\|b_j\| < \infty$ and such that

$$Tf = \sum_{j=1}^{\infty} \langle f,b_j\rangle a_j \quad \forall f \in U. \tag{1.1}$$

Sometimes these operators are referred to as trace class operators from U to H. It is well known that $L_1(U,H)$ is a Banach space with the norm

$$\|T\|_{L_1(U,H)} = \inf\Big\{ \sum_{j=1}^{\infty} \|a_j\|\|b_j\| : Tf = \sum_{j=1}^{\infty} \langle f,b_j\rangle a_j \ \forall f \in U \Big\}.$$

We note that $T \in L_1(U,H)$ is compact because (1.1) means that it can be approximated by operators of finite rank. Another characterization of $L_1(U,H)$ can be obtained via the polar decomposition of compact operators (see, for example, [6, Chapter 30] and [8, Chapter 7]).

Lemma 1.1. *Let $T \in L_1(H,H)$ and $\{e_k\}_{k\in\mathbf{N}}$ be an orthonormal basis for H. Then the trace of T,*

$$\mathrm{Tr}(T) = \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle, \tag{1.2}$$

exists and is independent of the choice of the orthonormal basis.

Proof. Since $T \in L_1(H,H)$ we have (1.1) for some $\{a_j\}_{j\in\mathbf{N}}, \{b_j\}_{j\in\mathbf{N}}$ in H. Then

$$\langle Te_k, e_k \rangle = \sum_{j=1}^{\infty} \langle e_k, b_j \rangle \langle a_j, e_k \rangle$$

and hence

$$\begin{aligned}\sum_{k=1}^{\infty} |\langle Te_k, e_k \rangle| &\le \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |\langle e_k, b_j \rangle \langle a_j, e_k \rangle| = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |\langle e_k, b_j \rangle \langle a_j, e_k \rangle| \\ &\le \sum_{j=1}^{\infty} \Big(\sum_{k=1}^{\infty} |\langle e_k, b_j \rangle|^2 \Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{\infty} |\langle a_j, e_k \rangle|^2 \Big)^{\frac{1}{2}} \le \sum_{j=1}^{\infty} \|a_j\| \|b_j\| < \infty.\end{aligned}$$

Therefore, the series in (1.2) converges absolutely and, by Fubini's theorem,

$$\begin{aligned}\sum_{k=1}^{\infty} \langle Te_k, e_k \rangle &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \langle e_k, b_j \rangle \langle a_j, e_k \rangle \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \langle e_k, b_j \rangle \langle a_j, e_k \rangle = \sum_{j=1}^{\infty} \langle a_j, b_j \rangle\end{aligned}$$

is independent of the orthonormal basis. □

Lemma 1.2. *If $T \in L_1(H_1,H_2)$, $S_1 \in L(H_2,H_3)$ and $S_2 \in L(H_3,H_1)$, then $S_1T \in L_1(H_1,H_3)$ and $TS_2 \in L_1(H_3,H_2)$. Moreover, if $T \in L_1(H_1,H_2)$, $S \in L(H_2,H_1)$, then $\mathrm{Tr}(ST) = \mathrm{Tr}(TS) \le \|S\| \|T\|_{L_1(H_1,H_2)}$. If $T \ge 0$, then $T \in L_1(H,H)$ if and only if the series in (1.2) converges for some orthonormal basis $\{e_k\}_{k\in\mathbf{N}}$ and in this case $\|T\|_{L_1(H,H)} = \mathrm{Tr}(T)$.*

Proof. The proofs for $H_i = H$ are given in [4, Appendix C]. The general cases are proved in the same way. $\square$

Definition 1.3 (Hilbert-Schmidt operator)**.** *An operator $T \in L(U,H)$ is Hilbert-Schmidt if $\sum_{k=1}^{\infty} \|Te_k\|^2 < \infty$ for an orthonormal basis $\{e_k\}_{k\in\mathbf{N}}$ of U.*

A straightforward calculation shows that if T is Hilbert-Schmidt, then the sum in Definition 1.3 is independent of the choice of the orthonormal basis. It is clear that Hilbert-Schmidt operators form a linear space denoted by $L_2(U,H)$ which, as Proposition 1.5 shows, becomes a Hilbert space with scalar product and norm

$$(1.3)\quad \langle T,S\rangle_{L_2(U,H)} = \sum_{k=1}^{\infty} \langle Te_k, Se_k\rangle_H,\ \|T\|_{L_2(U,H)} = \Big(\sum_{k=1}^{\infty} \|Te_k\|_H^2\Big)^{\frac{1}{2}},$$

where $\{e_k\}_{k\in\mathbf{N}}$ is any orthonormal basis of U.

Remark 1.4. We list a few facts about Hilbert Schmidt operators.

1. An operator $T \in L_2(U,H)$ if and only if $T^* \in L_2(H,U)$ and

$$\|T\|_{L_2(U,H)} = \|T^*\|_{L_2(H,U)}.$$

2. An operator $T \in L_2(U,H)$ if and only if $TT^* \in L_1(H,H)$ if and only if $T^*T \in L_1(U,U)$ and in this case

$$\|T\|^2_{L_2(U,H)} = \mathrm{Tr}(TT^*) = \mathrm{Tr}(T^*T).$$

3. If $T \in L_2(U,H)$ and $S \in L(U)$, then $TS \in L_2(U,H)$ and

$$\|TS\|_{L_2(U,H)} \le \|T\|_{L_2(U,H)}\|S\|_{L(U)}.$$

4. If $T \in L_2(U,H)$, then $\|T\|_{L(U,H)} \leq \|T\|_{L_2(U,H)}$.

The proofs are elementary and are left as an exercise.

Proposition 1.5. *The space $L_2(U,H)$ of Hilbert-Schmidt operators is a separable Hilbert space with scalar product and norm defined in* (1.3)*. If $\{f_k\}_{k\in\mathbf{N}}$ is an orthonormal basis of H and $\{e_k\}_{k\in\mathbf{N}}$ is an orthonormal basis of U, then the rank one operators $\{f_j \otimes e_k\}_{j,k\in\mathbf{N}}$ defined by $(f_j \otimes e_k)(u) := f_j\langle e_k, u\rangle$, $u \in U$, form an orthonormal basis for $L_2(U,H)$.*

Proof. We first prove completeness. Let $\{T_n\}_{n\in\mathbf{N}} \subset L_2(U,H)$ be a Cauchy sequence. Then $\{T_n\}_{n\in\mathbf{N}}$ is also a Cauchy sequence in $L(U,H)$ since $\|T\|_{L(U,H)} \leq \|T\|_{L_2(U,H)}$ for all $T \in L_2(U,H)$. Since $L(U,H)$ is complete there is $T \in L(U,H)$ such that

$$\|T_n - T\|_{L(U,H)} \to 0, \quad \text{as } n \to \infty.$$

Let $\varepsilon > 0$. If n is large enough, then by Fatou's lemma,

$$\begin{aligned}
\|T_n - T\|^2_{L_2(U,H)} &= \sum_{k=1}^{\infty} \|(T_n - T)e_k\|_H^2 \\
&= \sum_{k=1}^{\infty} \lim_{m\to\infty} \|(T_n - T_m)e_k\|_H^2 \\
&\leq \liminf_{m\to\infty} \sum_{k=1}^{\infty} \|(T_n - T_m)e_k\|_H^2 \\
&= \liminf_{m\to\infty} \|(T_n - T_m)\|^2_{L_2(U,H)} \leq \varepsilon.
\end{aligned}$$

This shows that $T_n \to T$ in $L_2(U,H)$ as $n \to \infty$ and that $T \in L_2(U,H)$. To show separability observe first that $f_j \otimes e_i \in L_2(U,H)$ for all $i, j \in \mathbf{N}$. If $T \in L_2(U,H)$, then

$$\langle f_j \otimes e_i, T\rangle_{L_2(U,H)} = \sum_{n=1}^{\infty} \langle e_i, e_n\rangle \langle f_j, Te_n\rangle = \langle f_j, Te_i\rangle. \tag{1.4}$$

By setting T equal to $f_k \otimes e_l$ in (1.4) it follows that $\{f_j \otimes e_i\}_{i,j\in\mathbf{N}}$ is an orthonormal system. To show that it is a complete system let $T \in L_2(U,H)$ and assume that $\langle f_j \otimes e_i, T\rangle_{L_2(U,H)} = 0$ for all $i,j \in \mathbf{N}$. Then $\langle f_j, Te_i\rangle = 0$ for all $i,j \in \mathbf{N}$ and thus $Te_i = 0$ for all $i \in \mathbf{N}$. Therefore, $T = 0$. □

The following proposition summarizes well-known results from the spectral theorem for self-adjoint compact linear operators on Hilbert space. For the proofs we refer to [6] and [8].

Proposition 1.6. *If $Q \in L(U)$, $Q \geq 0$, and $\mathrm{Tr}(Q) < \infty$, then there is an orthonormal basis $\{e_k\}_{k\in\mathbf{N}}$ of U such that $Qe_k = \lambda_k e_k$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq \lambda_{k+1} \geq \cdots \geq 0$, $\lambda_k \to 0$ as $k \to \infty$, and 0 is the only accumulation point of $\{\lambda_k\}_{k\in\mathbf{N}}$. Moreover,*

$$Qx = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k\rangle e_k, \quad x \in U.$$

1.2 Pseudo-inverse and the Cameron-Martin space

Let $T \in L(U,H)$ and define $\ker(T) = \{x \in U : Tx = 0\}$. Recall that T is one-to-one with inverse T^{-1} if and only if $\ker(T) \neq 0$. Since the restriction $T\big|_{(\ker(T))^\perp}$ is one-to-one we may define the pseudo-inverse of T, still denoted T^{-1}, by

$$T^{-1} = \left(T\big|_{(\ker(T))^\perp}\right)^{-1}$$

and thus T^{-1} is defined on the range of T,

$$T^{-1} : T(U) \to \big(\ker(T)\big)^\perp.$$

In the particular situation of Proposition 1.6 we have

$$Q^{-1}x = \sum_{\lambda_k>0}^{\infty} \lambda_k^{-1}\langle x, e_k\rangle e_k, \quad x \in U.$$

Let $Q \in L(U)$, $Q \geq 0$, and let $Q^{1/2} \in L(U)$ denote its unique positive square root, that is, $Q^{1/2} \geq 0$ and $Q^{1/2}Q^{1/2} = Q$. (Every positive operator in $L(U)$ has a unique positive square root, see [6, 31.2].) Let us introduce the *Cameron-Martin* space $U_0 = Q^{1/2}(U)$ with inner product

$$\langle u_0, v_0\rangle_0 = \langle Q^{-1/2}u_0, Q^{-1/2}v_0\rangle_U, \quad u_0, v_0 \in U_0,$$

where $Q^{-1/2}$ denotes the pseudo-inverse of $Q^{1/2}$ in case it is not one-to-one. Since

$$\|Q^{1/2}u\|_0^2 = \langle Q^{-1/2}Q^{1/2}u, Q^{-1/2}Q^{1/2}u\rangle_U = \|u\|_U^2, \quad u \in U,$$

it follows that

$$Q^{1/2} : \Big(\big(\ker(Q^{1/2})\big)^{\perp}, \langle\cdot,\cdot\rangle_U\Big) \to (U_0, \langle\cdot,\cdot\rangle_0)$$

is an isometric isomorphism. Hence, $(U_0, \langle\cdot,\cdot\rangle_0)$ is a separable Hilbert space. If $\{g_k\}_{k\in\mathbf{N}}$ is an orthonormal basis for $(\ker(Q^{1/2}))^{\perp}$, then it follows that $\{Q^{1/2}g_k\}_{k\in\mathbf{N}}$ is an orthonormal basis for $(U_0, \langle\cdot,\cdot\rangle_0)$. Let $L_2^0 = L_2(U_0, H)$ be the space of Hilbert-Schmidt operators from $U_0 \to H$ and let $L(U,H)_0 = \{T|_{U_0} : T \in L(U,H)\}$ denote the space of bounded linear operators $U \to H$ restricted to U_0. Notice that L_2^0 may contain unbounded operators $U \to H$. However, the next result shows that if $\mathrm{Tr}(Q) < \infty$, then $L(U,H)_0$ is dense in L_2^0.

Lemma 1.7. *There is an orthonormal basis of $L_2^0 = L_2(U_0, H)$ consisting of elements of $L(U,H)_0$. If $\mathrm{Tr}(Q) < \infty$, then $L(U,H)_0 \subset L_2^0$ and hence $L(U,H)_0$ is dense in L_2^0.*

Proof. Let $\{g_k\}_{k\in\mathbf{N}}$ be an orthonormal basis for $\big(\ker(Q)\big)^{\perp}$. Then, by the previous discussion, $\{Q^{1/2}g_k\}_{k\in\mathbf{N}}$ is an orthonormal basis of U_0. By Lemma 1.5 the set $\{f_j\otimes Q^{1/2}g_k\}_{j,k\in\mathbf{N}}$ is an orthonormal basis for $L_2(U_0,H)$, if $\{f_j\}_{j\in\mathbf{N}}$ is an orthonormal basis for H. This proves the first statement. To prove the second statement we complement $\{g_k\}_{k\in\mathbf{N}}$ to an orthonormal basis of U, still denoted by $\{g_k\}_{k\in\mathbf{N}}$, by adding an orthonormal basis of $\ker(Q^{1/2})$. Since $\mathrm{Tr}(Q)<\infty$, it follows that $Q^{1/2}\in L_2(U,U)$ by property (2) in Remark 1.4. If $T\in L(U,H)_0$, then, by property (3) in Remark 1.4,

$$\begin{aligned}\|T\|^2_{L_2^0} &= \sum_{k=1}^{\infty}\|TQ^{1/2}g_k\|^2 = \|TQ^{1/2}\|^2_{L_2(U,H)}\\ &\le \|T\|^2_{L(U,H)}\|Q^{1/2}\|^2_{L_2(U,U)} < \infty\end{aligned}$$

and thus $L(U,H)_0\subset L_2^0$. □

2 Elements of Banach space-valued stochastic analysis

Let $(U,\langle\cdot,\cdot\rangle_U)$ be a separable Hilbert space and let $(\Omega,\mathcal{F},P)$ be a probability space. In the present section we review some constructions and results from the theory of Banach space-valued stochastic analysis.

2.1 Infinite-dimensional Wiener processes

Let $\mathcal{B}(U)$ denote the Borel σ-algebra of U, that is, the smallest σ-algebra which contains all open subsets of U. Let μ be a probability measure on $(U,\mathcal{B}(U))$. By a real random variable on the probability space

$(U,\mathcal{B}(U),\mu)$ we understand a measurable function $X:(U,\mathcal{B}(U))\to(\mathbf{R},\mathcal{B}(\mathbf{R}))$, where $\mathcal{B}(\mathbf{R})$ is the real Borel σ-algebra. The law of X is the probability measure $\mu\circ X^{-1}$. For $v\in U$ let $v'\in U^*$ denote the functional given by $v'(u)=\langle v,u\rangle_U,\ u\in U$.

Definition 2.1. *A probability measure μ on $(U,\mathcal{B}(U))$ is Gaussian if for all $v\in U$, v' has a Gaussian law as a real-valued random variable on the probability space $(U,\mathcal{B}(U),\mu)$. That is, for all $v\in U$ there are $m_v\in\mathbf{R}$ and $\sigma_v\in\mathbf{R}_+$, such that, if $\sigma_v>0$,*

$$\left(\mu\circ(v')^{-1}\right)(A)=\mu\left(\{u\in U:v'(u)\in A\}\right)=\frac{1}{\sqrt{2\pi\sigma_v^2}}\int_A \mathrm{e}^{-\frac{(s-m_v)^2}{2\sigma_v^2}}\,\mathrm{d}s,$$

for all $A\in\mathcal{B}(\mathbf{R})$. If $\sigma_v=0$, then we require that $\mu\circ(v')^{-1}=\delta_{m_v}$, the Dirac measure concentrated at m_v.

We need the following lemma.

Lemma 2.2. *Let ν be a probability measure on $(U,\mathcal{B}(U))$ and $k\in\mathbf{N}$ be such that*

$$\int_U |\langle z,x\rangle|^k\,\mathrm{d}\nu(x)<\infty\quad\forall z\in U.$$

Then there is a constant $C(\nu,k)>0$ such that for all $h_1,\dots,h_k\in U$,

$$\int_U |\langle h_1,x\rangle\cdots\langle h_k,x\rangle|\,\mathrm{d}\nu(x)\le C(\nu,k)\|h_1\|\cdots\|h_k\|.$$

In particular, the symmetric k-form

$$(h_1,\dots,h_k)\mapsto\int_U\langle h_1,x\rangle\cdots\langle h_k,x\rangle\,\mathrm{d}\nu(x)$$

is continuous.

Proof. Let $n \in \mathbf{N}$ and define

$$U_n = \left\{ z \in U : \int_U \left|\langle z, x\rangle\right|^k \mathrm{d}\nu(x) \le n \right\}.$$

Then, by construction, $U = \bigcup_{n=1}^{\infty} U_n$. Notice that U_n is closed for all $n \in \mathbf{N}$. Indeed, let $z \in \overline{U}_n$ and take a sequence $U_n \ni z_j \to z$ as $j \to \infty$. Then $\left|\langle z_j, x\rangle\right|^k \to |\langle z, x\rangle|^k$ as $j \to \infty$ and thus, by Fatou's lemma,

$$\int_U |\langle z, x\rangle|^k \mathrm{d}\nu(x) \le \liminf_{j\to\infty} \int_U \left|\langle z_j, x\rangle\right|^k \mathrm{d}\nu(x) \le n,$$

so that $z \in U_n$.

Since U is a complete metric space, it follows from the Baire category theorem that there is n_0 such that U_{n_0} is not nowhere dense[1]. Therefore there are $r_0 > 0$ and $z_0 \in U_{n_0}$ such that the closed ball $B(z_0, r_0) \subset \overline{U}_{n_0} = U_{n_0}$. Therefore,

$$\int_U |\langle z_0 + y, x\rangle|^k \mathrm{d}\nu(x) \le n_0 \quad \forall y \in B(0, r_0),$$

and hence, for all $y \in B(0, r_0)$,

$$\begin{aligned} \int_U |\langle y, x\rangle|^k \mathrm{d}\nu(x) &= \int_U |\langle z_0 + y, x\rangle - \langle z_0, x\rangle|^k \mathrm{d}\nu(x) \\ &\le 2^{k-1} \int_U |\langle \underbrace{z_0 + y}_{\in U_{n_0}}, x\rangle|^k \mathrm{d}\nu(x) \\ &\quad + 2^{k-1} \int_U |\langle \underbrace{z_0}_{\in U_{n_0}}, x\rangle|^k \mathrm{d}\nu(x) \le 2^k n_0. \end{aligned} \tag{2.1}$$

Let $z \in U$ with $\|z\| = 1$ and $y = r_0 z$ so that $y \in B(0, r_0)$. By (2.1),

$$\int_U |\langle z, x\rangle|^k \mathrm{d}\nu(x) = r_0^{-k} \int_U |\langle y, x\rangle|^k \mathrm{d}\nu(x) \le 2^k n_0 r_0^{-k}.$$

[1] A set $A \subset U$ is nowhere dense, if $\overline{A}$ has empty interior.

Finally, if $h_1,\dots,h_k \in U\setminus\{0\}$, then by Hölder's inequality,

$$
\begin{aligned}
&\int_U \Big|\Big\langle \frac{h_1}{\|h_1\|},x\Big\rangle \cdots \Big\langle \frac{h_k}{\|h_k\|},x\Big\rangle\Big| \,\mathrm{d}\nu(x) \\
&\le \Big(\int_U \Big|\Big\langle \frac{h_1}{\|h_1\|},x\Big\rangle\Big|^k \mathrm{d}\nu(x)\Big)^{1/k} \cdots \Big(\int_U \Big|\Big\langle \frac{h_k}{\|h_k\|},x\Big\rangle\Big|^k \mathrm{d}\nu(x)\Big)^{1/k} \\
&\le 2^k n_0 r_0^{-k}.
\end{aligned}
$$

□

We next characterize Gaussian measures is terms of their Fourier transforms.

Theorem 2.3 (Characterization of Gaussian measure)**.** *A finite measure μ on $(U,\mathcal{B}(U))$ is Gaussian if and only if*

$$
\hat{\mu}(u) := \int_U \mathrm{e}^{\mathrm{i}\langle u,v\rangle_U}\,\mathrm{d}\mu(v) = \mathrm{e}^{\mathrm{i}\langle m,u\rangle_U - \frac{1}{2}\langle Qu,u\rangle_U},
$$

where $m \in U$ and $Q \in L(U), Q \ge 0$, with $\mathrm{Tr}(Q) < \infty$*. In this case we write $\mu = N(m,Q)$, and m and Q are called the mean and the covariance operator of μ. The measure μ is uniquely determined by m and Q.*

Proof. Assume that μ has Fourier transform

$$
\hat{\mu}(u) = \mathrm{e}^{\mathrm{i}\langle m,u\rangle - \frac{1}{2}\langle Qu,u\rangle}.
$$

We must check that for $v' \in U^*$ with $v'(u) = \langle v,u\rangle$ the measure $\mu_v = \mu\circ(v')^{-1}$ is Gaussian on $\mathbf{R}$. For $t \in \mathbf{R}$ we have, by assumption,

$$
\hat{\mu}(tv) = \mathrm{e}^{\mathrm{i}t\langle m,v\rangle - \frac{1}{2}t^2\langle Qv,v\rangle}.
$$

On the other hand, by the definition of the Fourier transform,

$$
\begin{aligned}
\hat{\mu}(tv) &= \int_U \mathrm{e}^{\mathrm{i}\langle tv,w\rangle}\,\mathrm{d}\mu(w) = \int_U \mathrm{e}^{\mathrm{i}t\langle v,w\rangle}\,\mathrm{d}\mu(w) \\
&= \int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}ts}\,\mathrm{d}\mu_v(s) = \hat{\mu}_v(t).
\end{aligned}
\tag{2.2}
$$

Therefore, $\hat{\mu}_v(t) = \mathrm{e}^{\mathrm{i}t\langle m,v\rangle - \frac{1}{2}t^2\langle Qv,v\rangle}$ and by uniqueness of the Fourier transform of finite measures on $\mathbf{R}$, the measure μ_v is Gaussian with mean $m_v = \langle m, v\rangle$ and covariance $\sigma_v^2 = \langle Qv, v\rangle$. The parameters m and Q determine μ uniquely by the uniqueness of the Fourier transform on U, see [4, p. 36].

Conversely, assume that μ is Gaussian on $(U, \mathcal{B}(U))$ as in Definition 2.1. Since,

$$\mathrm{d}\mu_v(s) = \frac{1}{\sqrt{2\pi\sigma_v^2}}\,\mathrm{e}^{-\frac{(s-m_v)^2}{2\sigma_v^2}}\,\mathrm{d}s \quad \text{or} \quad \mu_v = \delta_{m_v},$$

we have

$$\int_U |\langle x, v\rangle|\,\mathrm{d}\mu(x) = \int_U |v'(x)|\,\mathrm{d}\mu(x) = \int_{\mathbf{R}} |s|\,\mathrm{d}\mu_v(s) < \infty,$$

and it follows from Lemma 2.2 that

$$v \mapsto \int_U \langle x, v\rangle\,\mathrm{d}\mu(x)$$

is continuous. Thus, by the Riesz representation theorem there is a unique $m \in U$ such that

$$\int_U \langle x, v\rangle\,\mathrm{d}\mu(x) = \langle m, v\rangle.$$

Moreover, we have

$$\int_U |\langle x, v\rangle|^2\,\mathrm{d}\mu(x) = \int_{\mathbf{R}} |s|^2\,\mathrm{d}\mu_v(s) < \infty$$

and thus the bilinear form

$$(h_1, h_2) \mapsto \int_U \langle x, h_1\rangle\langle x, h_2\rangle\,\mathrm{d}\mu(x) - \langle m, h_1\rangle\langle m, h_2\rangle$$

is continuous by Lemma 2.2. Hence, by the Riesz representation theorem, there is a symmetric operator $Q \in L(U)$ such that

$$\langle Qh_1, h_2 \rangle = \int_U \langle x, h_1 \rangle \langle x, h_2 \rangle \, \mathrm{d}\mu(x) - \langle m, h_1 \rangle \langle m, h_2 \rangle.$$

Note that $Q \geq 0$ because

$$\begin{aligned} \langle Qh, h \rangle &= \int_U \langle x, h \rangle^2 \, \mathrm{d}\mu(x) - \langle m, h \rangle^2 \\ &= \int_U \langle x, h \rangle^2 \, \mathrm{d}\mu(x) - \Big(\int_U \langle x, h \rangle \, \mathrm{d}\mu(x) \Big)^2 \geq 0. \end{aligned}$$

In order to determine the Fourier transform of μ first note that

$$\langle m, v \rangle = \int_U \langle x, v \rangle \, \mathrm{d}\mu(x) = \int_{\mathbf{R}} s \, \mathrm{d}\mu_v(s) = m_v,$$

and

$$\begin{aligned} \langle Qv, v \rangle &= \int_U \langle x, v \rangle^2 \, \mathrm{d}\mu(x) - \Big(\int_U \langle x, v \rangle \, \mathrm{d}\mu(x) \Big)^2 \\ &= \int_{\mathbf{R}} s^2 \, \mathrm{d}\mu_v(s) - \Big(\int_{\mathbf{R}} s \, \mathrm{d}\mu_v(s) \Big)^2 = \sigma_v^2. \end{aligned}$$

Therefore, by (2.2) and the uniqueness of the Fourier transform on $\mathbf{R}$,

$$\hat{\mu}(v) = \hat{\mu}_v(1) = \mathrm{e}^{\mathrm{i} m_v - \frac{1}{2}\sigma_v^2} = \mathrm{e}^{\mathrm{i}\langle m, v \rangle - \frac{1}{2}\langle Qv, v \rangle}$$

as required.

Finally, we show that $\mathrm{Tr}(Q) < \infty$. Without loss of generality we may assume that $m = 0$. Otherwise, the translated measure $\tilde{\mu}(A) = \mu(A + m)$ has zero mean and the same covariance operator as μ. Let $c \in (0, \infty)$. Since $m = 0$, we have

$$\mathrm{e}^{-\frac{1}{2}\langle Qh, h \rangle} = \int_U \mathrm{e}^{\mathrm{i}\langle h, x \rangle} \, \mathrm{d}\mu(x) = \int_U \cos\langle h, x \rangle \, \mathrm{d}\mu(x).$$

Therefore, using that $1-\cos x \le \frac{1}{2}x^2$,

$$
\begin{aligned}
1-\mathrm{e}^{-\frac{1}{2}\langle Qh,h\rangle} &= \int_U (1-\cos\langle h,x\rangle)\,\mathrm{d}\mu(x) \\
&\le \frac{1}{2}\int_{\|x\|\le c} |\langle h,x\rangle|^2\,\mathrm{d}\mu(x) + 2\mu(\{x : \|x\| > c\}.
\end{aligned} \tag{2.3}
$$

Define $Q_c \in L(U)$, $Q_c \ge 0$, by

$$
\langle Q_c h_1, h_2\rangle = \int_{\|x\|\le c} \langle h_1, x\rangle\langle h_2, x\rangle\,\mathrm{d}\mu(x), \quad h_1,h_2 \in U.
$$

We have that $\mathrm{Tr}(Q_c) < \infty$, since

$$
\begin{aligned}
\mathrm{Tr}(Q_c) &= \sum_{k=1}^{\infty}\langle Q_c e_k, e_k\rangle = \sum_{k=1}^{\infty}\int_{\|x\|\le c}\langle e_k,x\rangle^2\,\mathrm{d}\mu(x) \\
&= \int_{\|x\|\le c}\sum_{k=1}^{\infty}\langle e_k,x\rangle^2\,\mathrm{d}\mu(x) \\
&= \int_{\|x\|\le c}\|x\|^2\,\mathrm{d}\mu(x) \le c^2 < \infty,
\end{aligned} \tag{2.4}
$$

where we used the monotone convergence theorem to interchange the sum and integral. We will show that there is $c > 0$ such that

$$
\langle Qh,h\rangle \le 2\log 4\langle Q_c h,h\rangle, \quad \forall h \in U, \tag{2.5}
$$

which implies that $\mathrm{Tr}(Q) \le 2\log 4\,\mathrm{Tr}(Q_c) < \infty$ in view of (2.4). Choose c such that

$$
\mu\{x \in U : \|x\| > c\} \le \frac{1}{8},
$$

and let $h \in U$ be such that $\langle Q_c h,h\rangle \le 1$. Then, (2.3) implies

$$
1-\mathrm{e}^{-\frac{1}{2}\langle Qh,h\rangle} \le \frac{1}{2}+\frac{1}{4} = \frac{3}{4},
$$

which yields $\langle Qh,h\rangle \le 2\log 4$. So if $h \in U$ is arbitrary but $\langle Q_c h,h\rangle \ne 0$, then we replace h by $\frac{h}{\sqrt{\langle Q_c h,h\rangle}}$ and deduce (2.5). On the other hand, if

$\langle Q_c h, h\rangle = 0$, then $\langle Q_c nh, nh\rangle = 0 \le 1$ for all $n \in \mathbf{N}$. Thus, $\langle Qh, h\rangle \le n^{-2} 2\log 4$. Since this is true for all $n \in \mathbf{N}$ it follows that $\langle Qh, h\rangle = 0$, which shows (2.5) in this case as well. □

Corollary 2.4. *Let μ be a Gaussian measure on U with mean m and covariance operator Q. Then, for all $u, v \in U$,*

$$\int_U \langle x, u\rangle_U \,\mathrm{d}\mu(x) = \langle m, u\rangle_U,$$
$$\int_U \langle x - m, u\rangle_U \langle x - m, v\rangle_U \,\mathrm{d}\mu(x) = \langle Qu, v\rangle_U,$$
$$\int_U \|x - m\|_U^2 \,\mathrm{d}\mu(x) = \mathrm{Tr}(Q).$$

Proof. The statement follows by inspecting the proof of Theorem 2.3 and is left to the reader as an exercise. □

Definition 2.5. *A U-valued random variable X on a probability space $(\Omega, \mathcal{F}, P)$, that is, a measurable mapping $X : (\Omega, \mathcal{F}, P) \to (U, \mathcal{B}(U))$, is Gaussian if the law $\mu = P \circ X^{-1}$ of X is a Gaussian measure on $(U, \mathcal{B}(U))$, that is, $P \circ X^{-1} = N(m, Q)$ for some $m \in U$ and $Q \in L(U)$. We call m the mean and Q the covariance operator of X.*

Proposition 2.6. *If X is a U-valued Gaussian random variable with mean m and covariance operator Q, then for all $u, v \in U$,*

$$\mathbf{E}(\langle X, u\rangle_U) = \langle m, u\rangle_U,$$
$$\mathbf{E}(\langle X - m, u\rangle_U \langle X - m, v\rangle_U) = \langle Qu, v\rangle_U,$$
$$\mathbf{E}(\|X - m\|_U^2) = \mathrm{Tr}(Q).$$

Proof. This follows from Corollary 2.4 by a change of variables. □

The following proposition gives a representation of a Gaussian random variable in terms the eigenpairs of its covariance operator, see Proposition 1.6.

Proposition 2.7. *Let $m \in U$ and $Q \in L(U)$, $Q \geq 0$, with* $\mathrm{Tr}(Q) < \infty$. *A U-valued random variable X on $(\Omega, \mathcal{F}, P)$ is Gaussian with $P \circ X^{-1} = N(m, Q)$ if and only if*

$$X = m + \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k, \tag{2.6}$$

where (λ_k, e_k) are the eigenpairs of Q and β_k are independent real random variables with $P \circ \beta_k^{-1} = N(0,1)$ if $\lambda_k > 0$ and $\beta_k = 0$ otherwise. The series in (2.6) *converges in $L_2(\Omega, \mathcal{F}, P; U)$.*

Proof. Let X be Gaussian with $P \circ X^{-1} = N(m, Q)$. Since $\{e_k\}_{k \in \mathbf{N}}$ is an orthonormal basis for U, it follows that, for fixed $\omega \in \Omega$,

$$X(\omega) = \sum_{k=1}^{\infty} \langle X(\omega), e_k \rangle e_k.$$

Since X is Gaussian we have that $\langle X, e_k \rangle$ is a real Gaussian random variable. By Proposition 2.6 we have

$$\begin{aligned}
&\mathbf{E}(\langle X, e_k \rangle) = \langle m, e_k \rangle, \\
&\mathbf{E}\left(\langle X - m, e_k \rangle \langle X - m, e_l \rangle\right) = \langle Q e_k, e_l \rangle = \lambda_k \delta_{kl}.
\end{aligned}$$

Define

$$\beta_k = \begin{cases} \lambda_k^{-\frac{1}{2}} \langle X - m, e_k \rangle, & \text{if } \lambda_k > 0, \\ 0, & \text{if } \lambda_k = 0. \end{cases}$$

If $\lambda_k > 0$, then β_k is a Gaussian random variable with $P \circ \beta_k^{-1} = N(0,1)$ and $X = m + \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k$. It remains to show that the random variables $\{\beta_k\}$ are independent. We will use the well-known fact that if

$Y = (Y_1, \ldots, Y_n)$ is an $\mathbf{R}^n$-valued Gaussian random variable, then the family $\{Y_k\}_{k=1}^n$, of real random variables are independent if and only if for $k \neq l$, $\mathbf{E}(Y_k Y_l) = 0$. Here, $\beta = (\beta_1, \ldots, \beta_n)$, where $n \in \mathbf{N}$ is fixed but arbitrary, is an $\mathbf{R}^n$-valued Gaussian random variable. Indeed, since X is Gaussian it follows that, for any $v \in \mathbf{R}^n$,

$$\langle \beta, v \rangle_{\mathbf{R}^n} = \sum_{k=1}^{n} v_k \beta_k = \sum_{\lambda_k > 0} v_k \lambda_k^{-\frac{1}{2}} \langle X - m, e_k \rangle_U = \Big\langle X, \sum_{\lambda_k > 0} v_k \lambda_k^{-\frac{1}{2}} e_k \Big\rangle_U + C$$

is real Gaussian and, hence, that β is $\mathbf{R}^n$-valued Gaussian. Moreover, $\mathbf{E}(\beta_k \beta_l) = \delta_{kl}$ for $k \neq l$ so that $\beta_1, \ldots, \beta_n$ are independent.

Finally, the series in (2.6) converges in $L_2(\Omega, \mathcal{F}, P; U)$, since by Parseval's identity and the fact that $\sum_{k=1}^{\infty} \lambda_k = \mathrm{Tr}(Q) < \infty$,

$$\begin{aligned}
\Big\| \sum_{k=n}^{m} \lambda_k^{-\frac{1}{2}} \beta_k e_k \Big\|_{L_2(\Omega, \mathcal{F}, P; U)}^2 &= \int_{\Omega} \Big\| \sum_{k=n}^{m} \lambda_k^{-\frac{1}{2}} \beta_k e_k \Big\|_U^2 \, \mathrm{d}P \\
&= \mathbf{E}\Big(\Big\| \sum_{k=n}^{m} \lambda_k^{-\frac{1}{2}} \beta_k e_k \Big\|_U^2 \Big) = \mathbf{E}\Big(\sum_{k=n}^{m} \lambda_k \beta_k^2 \Big) \\
&= \sum_{k=n}^{m} \lambda_k \mathbf{E}\big(\beta_k^2\big) = \sum_{k=n}^{m} \lambda_k \to 0 \text{ as } m, n \to \infty.
\end{aligned}$$

Conversely, let β_k, e_k and λ_k be as assumed. Define

$$X = m + \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k,$$

which converges in $L_2(\Omega, \mathcal{F}, P; U)$ by the above computation. We have to show that X is a Gaussian random variable with mean m and covariance operator Q. If $u \in U$, then

$$\Big\langle m + \sum_{k=1}^{n} \sqrt{\lambda_k} \beta_k e_k, u \Big\rangle = \langle m, u \rangle + \sum_{k=1}^{n} \sqrt{\lambda_k} \beta_k \langle e_k, u \rangle \tag{2.7}$$

is real a real Gaussian random variable since $\beta_1, \ldots, \beta_n$ are independent Gaussian random variables. Moreover, the series on the right side of (2.7) converges in $L_2(\Omega, \mathcal{F}, P; \mathbf{R})$ and hence its limit $\langle X, u \rangle$ is real Gaussian. Therefore, X is a Gaussian random variable. Finally, by the assumption on $\{\beta_k\}$, for the mean we obtain $\mathbf{E}(\langle X, u \rangle) = \langle m, u \rangle$ and for the covariance we have

$$
\begin{aligned}
\mathbf{E}(\langle X - m, u \rangle \langle X - m, v \rangle) &= \mathbf{E}\Big(\Big\langle \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k, u \Big\rangle \Big\langle \sum_{l=1}^{\infty} \sqrt{\lambda_l} \beta_l e_l, v \Big\rangle \Big) \\
&= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sqrt{\lambda_k \lambda_l} \mathbf{E}(\beta_k \beta_l) \langle e_k, u \rangle \langle e_l, v \rangle \\
&= \sum_{k=1}^{\infty} \lambda_k \langle e_k, u \rangle \langle e_k, v \rangle = \langle Qu, v \rangle .
\end{aligned}
$$

□

Corollary 2.8 (Existence of Gaussian measures). *For each $m \in U$ and $Q \in L(U)$, $Q \geq 0$, with $\mathrm{Tr}(Q) < \infty$, there exists $\mu = N(m, Q)$.*

Proof. For the given m and Q, construct a Gaussian random variable X according to Proposition 2.7 and take $\mu = P \circ X^{-1}$. □

Remark 2.9. In the above construction we assumed that there exist a probability space with a countably infinite family of independent real Gaussian random variables. This is a nontrivial fact from probability theory.

Definition 2.10. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $I \subset \mathbf{R}$ be an interval (possibly infinite). A U-valued stochastic process $\{X(t)\}_{t \in I}$ is a set of U-valued random variables $X(t)$ on $(\Omega, \mathcal{F}, P)$ where $t \in I$. Two* *stochastic processes* $\{X(t)\}_{t\in I}$ *and* $\{Y(t)\}_{t\in I}$ *are versions (or modifications) of each other, if*

$$P(\{X(t) \neq Y(t)\}) = 0, \quad \textit{for all } t \in I.$$

They are indistinguishable (or indistinguishable versions of each other), if

$$P\Big(\bigcup_{t\in I}\{X(t) \neq Y(t)\}\Big) = 0.$$

Since I is uncountable, being indistinguishable is much stronger than being versions. This is because the exceptional null sets, where the processes do not coincide, may depend on t, with t ranging in an uncountable set in case two processes are only versions of each other.

Definition 2.11. *A* U*-valued stochastic process* $\{W(t)\}_{t\geq 0}$ *is called a (nuclear) Q-Wiener process if*

1. $W(0) = 0$*;*

2. $\{W(t)\}_{t\geq 0}$ *has continuous paths almost surely, that is, the mapping* $t \mapsto W(t, \omega)$ *is continuous for almost every* $\omega \in \Omega$*;*

3. $\{W(t)\}_{t\geq 0}$ *has independent increments, that is, for any finite partition* $0 = t_0 \leq t_1 \leq \cdots \leq t_{m-1} \leq t_m < \infty$ *the random variables* $W(t_1), W(t_2) - W(t_1), \cdots, W(t_m) - W(t_{m-1})$*, are independent;*

4. *the increments have Gaussian laws, more precisely,*

$$P \circ (W(t) - W(s))^{-1} = N(0, (t-s)Q), \quad 0 \leq s \leq t.$$

It follows from the definition that $\mathrm{Tr}(Q) < \infty$ and that we may as well assume that $\{W(t)\}_{t\geq 0}$ has continuous paths for all $\omega \in \Omega$ by re-defining $W(t,\omega) = 0$ for those ω where $t \mapsto W(t,\omega)$ is not continuous.

Proposition 2.12 (Representation of Q-Wiener process)**.** *Let $Q \in L(U)$, $Q \geq 0$, with $\mathrm{Tr}(Q) < \infty$. A U-valued process $\{W(t)\}_{t\geq 0}$ is a U-valued Q-Wiener process if and only if*

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k, \tag{2.8}$$

where (λ_k, e_k) are the eigenpairs of Q and $\{\beta_k(t)\}_{t\geq 0}$ are independent real-valued standard Brownian motions on $(\Omega,\mathcal{F},P)$. For each $T > 0$, the series in (2.8) *converges in $L_2\big(\Omega,\mathcal{F},P;C([0,T],U)\big)$. In particular, for every $Q \in L(U)$ with $Q \geq 0$ and $\mathrm{Tr}(Q) < \infty$, there exists a Q-Wiener process.*

Proof. Let $\{W(t)\}_{t\geq 0}$ be a Q-Wiener process. Since

$$P \circ W(t)^{-1} = N(0,tQ),$$

it follows, as in the proof of Proposition 2.7, that

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k,$$

where

$$\beta_k(t) = \begin{cases} \lambda_k^{-\frac{1}{2}}\langle W(t), e_k\rangle, & \text{if } \lambda_k > 0, \\ 0, & \text{if } \lambda_k = 0, \end{cases}$$

and the sum converges in $L_2(\Omega,\mathcal{F},P;U)$. Also, $P \circ \beta_k(t)^{-1} = N(0,t)$ and the random variables $\{\beta_k(t)\}_{k\in\mathbf{N}}$ are independent, for fixed t. We have to show that, in fact, $\{\beta_k(\cdot)\}_{k\in\mathbf{N}}$ is a family of independent standard real Brownian motions. Since

$$(2.9)\quad \beta_k(t_n)-\beta_k(t_{n-1})=\lambda_k^{1/2}\langle W(t_n)-W(t_{n-1}),e_k\rangle,\ 0\le t_{n-1}\le t_n,$$

for all $k\in\mathbf{N}$, it follows that $\beta_k(0)=0$, $\{\beta_k(t)\}_{t\ge 0}$ has continuous paths almost surely, $\{\beta_k(t)\}_{t\ge 0}$ has independent increments and $P\circ\big(\beta_k(t)-\beta_k(s)\big)^{-1}=N(0,t-s)$ for $t\ge s$. It remains to show that $\{\beta_k(\cdot)\}_{k\in\mathbf{N}}$ is a family of independent stochastic processes. Take $\{k_i\}_{i=1}^n\subset\mathbf{N}$ distinct and $0=t_0\le t_1\le\cdots\le t_m<\infty$. We must show that the σ-algebras

$$\sigma\big(\beta_{k_1}(t_1),\dots,\beta_{k_1}(t_m)\big),\dots,\sigma\big(\beta_{k_n}(t_1),\dots,\beta_{k_n}(t_m)\big)$$

are independent. We proceed by induction on m. For $m=1$, the random variables $\{\beta_{k_i}(t_1)\}_{i=1}^n$ are independent as observed before. Now take $0=t_0\le t_1\le\cdots\le t_{m+1}$ and assume that

$$\sigma\big(\beta_{k_1}(t_1),\dots,\beta_{k_1}(t_m)\big),\dots,\sigma\big(\beta_{k_n}(t_1),\dots,\beta_{k_n}(t_m)\big)$$

are independent. Note first that

$$\sigma\big(\beta_{k_i}(t_1),\dots,\beta_{k_i}(t_{m+1})\big)=\sigma\big(\beta_{k_i}(t_1),\dots,\beta_{k_i}(t_m),\beta_{k_i}(t_{m+1})-\beta_{k_i}(t_m)\big).$$

Also, (2.9) and the fact that $W(t_n)-W(t_{n-1})$ is Gaussian imply, as in the proof of Proposition 2.7, that the random variables $\{\beta_k(t_n)-\beta_k(t_{n-1})\}_{n=2}^{m+1}$ are independent. Then, for $A_{ij}\in\mathcal{B}(\mathbf{R})$, $i=1,\dots,n$, $j=$

$1, \dots, m$,

$$
\begin{aligned}
&P\Big(\bigcap_{i=1}^{n}\Big\{\beta_{k_i}(t_1) \in A_{i,1}, \dots, \beta_{k_i}(t_m) \in A_{i,m}, \beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) \in A_{i,m+1}\Big\}\Big) \\
&= P\Big(\bigcap_{i=1}^{n}\bigcap_{j=1}^{m}\Big\{\beta_{k_i}(t_j) \in A_{i,j}\Big\}\bigcap_{i=1}^{n}\Big\{\beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) \in A_{i,m+1}\Big\}\Big) \\
&= P\Big(\bigcap_{i=1}^{n}\bigcap_{j=1}^{m}\Big\{\beta_{k_i}(t_j) \in A_{i,j}\Big\}\Big)P\Big(\bigcap_{i=1}^{n}\Big\{\beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) \in A_{i,m+1}\Big\}\Big) \\
&= \prod_{i=1}^{n}P\Big(\bigcap_{j=1}^{m}\Big\{\beta_{k_i}(t_j) \in A_{i,j}\Big\}\Big)\prod_{i=1}^{n}P\Big(\Big\{\beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) \in A_{i,m+1}\Big\}\Big) \\
&= \prod_{i=1}^{n}P\Big(\bigcap_{j=1}^{m}\Big\{\beta_{k_i}(t_j) \in A_{i,j}\Big\}\Big)P\Big(\Big\{\beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) \in A_{i,m+1}\Big\}\Big) \\
&= \prod_{i=1}^{n}P\Big(\bigcap_{j=1}^{m}\Big\{\beta_{k_i}(t_j) \in A_{i,j}\Big\}\cap\Big\{\beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) \in A_{i,m+1}\Big\}\Big),
\end{aligned}
$$

which finishes the proof of the induction step and hence the induction.

Conversely, let $\{\beta_k(\cdot)\}_{k\in\mathbf{N}}$ and Q be given as in the statement of the theorem. Define

$$
W(t) = \sum_{k=1}^{n} \sqrt{\lambda_k}\beta_k(t)e_k, \tag{2.10}
$$

where for fixed t is the series converges in $L_2(\Omega, \mathcal{F}, P; U)$ as Q has finite trace. It is straightforward to check that $W(0) = 0$, $\{W(t)\}_{t\geq 0}$ has independent Gaussian increments with the required covariance operator (compare with the proof of Proposition 2.7). The almost sure continuity of the paths will follow from the $L_2\big(\Omega, \mathcal{F}, P; C([0,T], U)\big)$ convergence of (2.10). In order to prove this we recall that Doob's maximal inequality states that if $\{M(t)\}_{t\geq 0}$ is a real-valued martingale, then

$$\Big(\mathbf{E}\Big(\sup_{0\leq t\leq T}|M(t)|^p\Big)\Big)^{\frac{1}{p}} \leq \frac{p}{p-1}\Big(\mathbf{E}\big(|M(T)|^p\big)\Big)^{\frac{1}{p}}, \quad 1<p<\infty.$$

It is well known that a real-valued Brownian motion is a martingale (with respect to itself) and therefore

$$\begin{aligned}&\mathbf{E}\Big(\sup_{0\leq t\leq T}\Big\|\sum_{k=n}^{m}\sqrt{\lambda_k}\beta_k(t)e_k\Big\|_U^2\Big) = \mathbf{E}\Big(\sup_{0\leq t\leq T}\sum_{k=n}^{m}\lambda_k\beta_k(t)^2\Big)\\ &\leq \sum_{k=n}^{m}\lambda_k\mathbf{E}\Big(\sup_{0\leq t\leq T}\beta_k(t)^2\Big) \leq 4\sum_{k=n}^{m}\lambda_k\mathbf{E}\big(\beta_k(T)^2\big) = 4T\sum_{k=n}^{m}\lambda_k \to 0,\end{aligned}$$

as $n\to\infty$. The fact that the space $L_2\big(\Omega,\mathcal{F},P;C([0,T],U)\big)$ is complete finishes the proof. □

Remark 2.13. As in the proof of the existence of a Gaussian measure, we used here the nontrivial fact that there is a probability space with a countably infinite set of independent Brownian motions. Even the existence of a single Brownian motion is far from obvious. We refer to the standard literature on probability theory.

2.2 Wiener processes with respect to a filtration

We start with a few definitions.

Definition 2.14. *A filtration is a family of* σ*-algebras* $\{\mathcal{F}_t\}_{t\geq 0}$ *with* $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ *for* $t\leq s$. *A filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *on* $(\Omega,\mathcal{F},P)$ *is called normal if*

- $\mathcal{F}_0$ *contains all sets* $A\in\mathcal{F}$ *such that* $P(A)=0$*;*
- $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t}\mathcal{F}_s, \quad \forall t\in[0,T].$

Definition 2.15. *A Q-Wiener process* $\{W(t)\}_{t\geq 0}$ *is called a Q-Wiener process with respect to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *if*

- $\{W(t)\}_{t\geq 0}$ *is adapted to* $\{\mathcal{F}_t\}_{t\geq 0}$*, that is,* $W(t)$ *is* $\mathcal{F}_t$ *measurable for all* $t \geq 0$*;*
- *the random variable* $W(t) - W(s)$ *is independent of* $\mathcal{F}_s$ *for all fixed* $s \in [0,t]$.

To see that for a given Q-Wiener process $\{W(t)\}_{t\geq 0}$ there is always a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\{W(t)\}_{t\geq 0}$ becomes a Q-Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, define

$$\mathcal{N} := \{A \in \mathcal{F} : P(A) = 0\}, \quad \tilde{\mathcal{F}}_s := \sigma(W(r) : r \leq s), \quad \tilde{\mathcal{F}}_s^0 := \sigma(\mathcal{N} \cup \tilde{\mathcal{F}}_s)$$

and

$$\mathcal{F}_s := \bigcap_{r>s} \tilde{\mathcal{F}}_s^0. \tag{2.11}$$

Proposition 2.16. *If* $\{W(t)\}_{t\geq 0}$ *is a U-valued Q-Wiener process on the measure space* $(\Omega, \mathcal{F}, P)$*, then* $\{W(t)\}_{t\geq 0}$ *is a Q-Wiener process with respect to the normal filtration defined in* (2.11).

Proof. That $\{W(t)\}_{t\geq 0}$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ is clear from the construction. Thus, we only need to show that $W(t) - W(s)$ independent of $\mathcal{F}_s$ for all fixed $s \in [0,t]$. We first show that $W(t) - W(s)$ is independent of $\tilde{\mathcal{F}}_s$. Fix $0 \leq s \leq t$ and take $0 \leq t_1 < t_2 < \cdots < t_n \leq s$. Then

$$\sigma\big(W(t_1), \ldots, W(t_n)\big) = \sigma\big(W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})\big)$$

is independent of $W(t)-W(s)$ as $\{W(t)\}_{t\geq 0}$ has independent increments. Then $W(t)-W(s)$ is also independent of $\tilde{\mathcal{F}}_s^0$. Finally, we show that $W(t)-W(s)$ is independent of $\mathcal{F}_s$. By the continuity of paths,

$$W(t)-W(s)=\lim_{n\to\infty}\big(W(t)-W(s+\tfrac{1}{n})\big).$$

If n is large enough (such that $s+\frac{1}{n}\leq t$), then $W(t)-W(s+\frac{1}{n})$ is independent of $\tilde{\mathcal{F}}^0_{s+\frac{1}{n}}\supset\mathcal{F}_s$ and hence of $\mathcal{F}_s$. Therefore, $W(t)-W(s)$ is also independent of $\mathcal{F}_s$. □

2.3 Martingales in Banach space

Let E be a Banach space. An E-valued random variable X on $(\Omega,\mathcal{F},P)$ is *Bochner integrable* if $X:(\Omega,\mathcal{F},P)\to(E,\mathcal{B}(E))$ is measurable and $\int_\Omega \|X(\omega)\|\,\mathrm{d}P(\omega)<\infty$. A Banach space E is called *separable* if there is a countable dense subset of E. If E is separable then X is measurable if and only if $l(X):(\Omega,\mathcal{F},P)\to(\mathbf{R},\mathcal{B}(\mathbf{R}))$ is measurable for all $l\in E^*$ as shown in Corollary 2.19 below.

Lemma 2.17. *Let E be a separable Banach space. Then there is a countable subset $\{l_n\}_{n\in\mathbf{N}}\subset E^*$ such that for all $x,y\in E$, $x\neq y$, there is $l_{n^*}\in\{l_n\}_{n\in\mathbf{N}}$ with $l_{n^*}(x)\neq l_{n^*}(y)$, that is, $\{l_n\}_{n\in\mathbf{N}}$ separates the points of E. Moreover, $\|x\|=\sup_{n\in\mathbf{N}}l_n(x)$.*

Proof. Since E is separable, there is $\{x_n\}_{n\in\mathbf{N}}\subset E$ such that $\{x_n\}_{n\in\mathbf{N}}$ is dense in E. By the Hahn-Banach theorem, there is $\{l_n\}_{n\in\mathbf{N}}\subset E^*$ such that $l_n(x_n)=\|x_n\|$ and $\|l_n\|=1$. If $x\in E$ is arbitrary, then there is a sequence $\{x_{n_k}\}_{k\in\mathbf{N}}\subset\{x_n\}_{n\in\mathbf{N}}$ such that $x_{n_k}\to x$ as $k\to\infty$. We have

$$l_{n_k}(x)=l_{n_k}(-x_{n_k}+x)+l_{n_k}(x_{n_k})=l_{n_k}(x-x_{n_k})+\|x_{n_k}\|.$$

Since $\|x_{n_k}\| \to \|x\|$ and $|l_{n_k}(x-x_{n_k})| \le \|l_{n_k}\|_{E^*}\|x-x_{n_k}\| = \|x-x_{n_k}\| \to 0$ as $k \to \infty$, we conclude that $l_{n_k}(x) \to \|x\|$ as $k \to \infty$. It follows that there is n^* such that $l_{n^*}(x) > 0$ if $x \neq 0$. Thus, $\{l_n\}$ separates the points of E. Finally, since $l_n(x) \le \|x\|$ for all $x \in E$, $n \in \mathbf{N}$, and $l_{n_k}(x) \to \|x\|$ as $k \to \infty$, it follows that $\|x\| = \sup_{n\in\mathbf{N}} l_n(x)$ for all $x \in E$. $\square$

Lemma 2.18. *If E is a separable Banach space and*

$$\mathcal{C} := \Big\{\{x \in E : l(x) \le \alpha\}\Big\}_{\alpha\in\mathbf{R},\ l\in E^*}, \tag{2.12}$$

then $\sigma(\mathcal{C}) = \mathcal{B}(E)$.

Proof. By Lemma 2.17, there is $\{l_n\}_{n\in\mathbf{N}} \subset E^*$ with $\|x\| = \sup_{n\in\mathbf{N}} l_n(x)$ for all $x \in E$. Let $a \in E$ and $r > 0$, and denote the open ball centered at a with radius r by $B(a,r)$. Then,

$$\begin{aligned} B(a,r) &= \bigcup_{m=1}^{\infty} \overline{B}\big(a, r(1-\tfrac{1}{m})\big) \\ &= \bigcup_{m=1}^{\infty} \big\{x \in E : \|x-a\| \le r(1-\tfrac{1}{m})\big\} \\ &= \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty} \{x \in E : l_n(x-a) \le r(1-\tfrac{1}{m})\}. \end{aligned}$$

Thus $\sigma(\mathcal{C})$ contains open balls of E and therefore, $\mathcal{B}(E) \subset \sigma(\mathcal{C})$. But $\sigma(\mathcal{C}) \subset \mathcal{B}(E)$, since all $l \in E^*$ are continuous. Thus, $\mathcal{B}(E) = \sigma(\mathcal{C})$. $\square$

Corollary 2.19 (Weak versus strong measurability)**.** *If E is a separable Banach space and $(\Omega, \mathcal{F}, P)$ is a measure space, then X is an E-valued random variable if and only if $l(X)$ is an $\mathbf{R}$-valued random variable for all $l \in E^*$. In other words: X is strongly measurable if and only if it is weakly measurable.*

Proof. If X is measurable, then $l(X)$ is measurable for all $l \in E^*$, since l is continuous. Conversely, since $\mathcal{B}(E) = \sigma(\mathcal{C})$, where $\mathcal{C}$ is defined in (2.12), it is enough to show that $X^{-1}(C) \in \mathcal{F}$ for all $C \in \mathcal{C}$. Let $l \in E^*$ and take a typical set $C = \{x \in E : l(x) \leq \alpha\}$ from $\mathcal{C}$. Since $l(X)$ is measurable, we have

$$X^{-1}(C) = \{\omega \in \Omega : X(\omega) \in C\} = \{\omega \in \Omega : l(X(\omega)) \leq \alpha\} \in \mathcal{F}.$$

□

Proposition 2.20 (Conditional expectation)**.** *Let E be a real separable Banach space, let X be an E-valued Bochner integrable random variable on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. Then there is a unique, up to a set of P-measure 0, Bochner integrable E-valued $\mathcal{G}$-measurable random variable Z such that*

$$\int_A X \, \mathrm{d}P = \int_A Z \, \mathrm{d}P \quad \text{for all } A \in \mathcal{G}. \tag{2.13}$$

The random variable Z is called the conditional expectation of X given $\mathcal{G}$ and is denoted by $Z = \mathbf{E}(X|\mathcal{G})$. Furthermore,

$$\|\mathbf{E}(X|\mathcal{G})\| \leq \mathbf{E}\big(\|X\| \big| \mathcal{G}\big) \quad P\text{-a.e.} \tag{2.14}$$

Proof. We first prove uniqueness. Suppose that Z_1, Z_2 are Bochner integrable and $\mathcal{G}$-measurable E-valued random variables such that

$$\int_A X \, \mathrm{d}P = \int_A Z_1 \, \mathrm{d}P = \int_A Z_2 \, \mathrm{d}P \quad \forall A \in \mathcal{G}.$$

Since E is separable, there is a subset $\{l_n\}_{n \in \mathbf{N}}$ that separates the points of E. For all $n \in \mathbf{N}$ and $A \in \mathcal{G}$ we have

$$l_n\Big(\int_A Z_1 \, \mathrm{d}P\Big) = l_n\Big(\int_A Z_2 \, \mathrm{d}P\Big),$$

which implies, by the continuity of l_n, that

$$\int_A l_n(Z_1)\,\mathrm{d}P = \int_A l_n(Z_2)\,\mathrm{d}P.$$

Therefore,

$$\int_A \big(l_n(Z_1) - l_n(Z_2)\big)\,\mathrm{d}P = 0, \quad \forall n \in \mathbf{N},\ A \in \mathcal{G}.$$

By taking first A to be $\{\omega \in \Omega : l_n(Z_1(\omega)) > l_n(Z_2(\omega))\}$, then to be $\{\omega \in \Omega : l_n(Z_1(\omega)) < l_n(Z_2(\omega))\}$ (both of these sets belong to $\mathcal{G}$), it follows that $l_n(Z_1) = l_n(Z_2)$, P-a.e. Therefore, the set

$$\Omega_0 = \bigcap_{n\in\mathbf{N}} \{\omega \in \Omega :\ l_n(Z_1(\omega)) = l_n(Z_2(\omega))\}$$

satisfies $P(\Omega_0) = 1$. If $\omega \in \Omega_0$, then $l_n(Z_1(\omega)) = l_n(Z_2(\omega))$ for all $n \in \mathbf{N}$. But this is only possible if $Z_1(\omega) = Z_2(\omega)$ for all $\omega \in \Omega_0$ as $\{l_n\}_{n\in\mathbf{N}}$ separates points. This finishes the proof the uniqueness part of the statement.

Next we show existence. Assume first that X is a simple random variable, that is, there are $X_1,\dots,X_N \in E$ and disjoint sets $A_1,\dots,A_n \in \mathcal{F}$ such that

$$X = \sum_{k=1}^{N} X_k 1_{A_k}$$

and define

$$Z = \sum_{k=1}^{N} X_k \mathbf{E}(1_{A_k}|\mathcal{G}). \tag{2.15}$$

It is clear from the definition, by the properties of the conditional expectation of real random variables, that Z is $\mathcal{G}$-measurable and that $\int_A Z\,\mathrm{d}P = \int_A X\,\mathrm{d}P$ for all $A \in \mathcal{G}$. Moreover,

$$\|Z\| \le \sum_{k=1}^{N} \|X_k\| \mathbf{E}(1_{A_k}|\mathcal{G}) = \mathbf{E}\Big(\sum_{k=1}^{N} \|X_k\| 1_{A_k}\Big|\mathcal{G}\Big) = \mathbf{E}\big(\|X\|\,\big|\,\mathcal{G}\big).$$

Taking expectations and using the law of double expectation for real random variables, we get

$$\mathbf{E}(\|Z\|) \leq \mathbf{E}\Big(\mathbf{E}\big(\|X\|\,\big|\,\mathcal{G}\big)\Big) = \mathbf{E}(\|X\|). \tag{2.16}$$

Let X be a general E-valued Bochner-integrable random variable. Then then there is a sequence of simple functions X_n such that $\|X_n(\omega) - X(\omega)\| \to 0$ as $n \to 0$ in a decreasing way, see, for example, [4, Lemma 1.1]. By Lebesgue's dominated convergence theorem this also holds in $L_1(\Omega, \mathcal{F}, P; E)$. Define Z_n as in (2.15), replacing X by X_n. Then, by (2.16), for all $m, n \in \mathbf{N}$,

$$\mathbf{E}(\|Z_n - Z_m\|) \leq \mathbf{E}(\|X_n - X_m\|).$$

Thus, $\{Z_n\}$ is a Cauchy sequence in $L_1(\Omega, \mathcal{G}, P; E)$ as $\{X_n\}$ is a Cauchy sequence in $L_1(\Omega, \mathcal{F}, P; E)$. Since $L_1(\Omega, \mathcal{G}, P; E)$ is complete, there exists $Z \in L_1(\Omega, \mathcal{G}, P; E)$ such that $Z_n \to Z$ in $L_1(\Omega, \mathcal{G}, P; E)$ and, in particular, Z is $\mathcal{G}$-measurable. Then, for all $A \in \mathcal{G}$,

$$\int_A X \,\mathrm{d}P = \int_A \lim_{n\to\infty} X_n \,\mathrm{d}P = \lim_{n\to\infty} \int_A X_n \,\mathrm{d}P = \lim_{n\to\infty} \int_A Z_n \,\mathrm{d}P = \int_A Z \,\mathrm{d}P.$$

Finally, since $Z_n \to Z$ as $n \to \infty$ in $L_1(\Omega, \mathcal{G}, P; E)$, it follows that there is a subsequence $\{Z_{n_k}\}$ of $\{Z_n\}$ which converges P-a.e. to Z. Clearly, the corresponding subsequence $X_{n_k} \to X$ as $k \to \infty$ both in $L_1(\Omega, \mathcal{F}, P; E)$ and pointwise for all $\omega \in \Omega$. Therefore, P-a.e.,

$$\|\mathbf{E}(X|\mathcal{G})\| = \|Z\| = \lim_{k\to\infty} \|Z_{n_k}\| \leq \lim_{k\to\infty} \mathbf{E}\big(\|X_{n_k}\|\,\big|\,\mathcal{G}\big) = \mathbf{E}\big(\|X\|\,\big|\,\mathcal{G}\big).$$

□

For later reference we note the ”law of double expectation”

$$\mathbf{E}\Big(\mathbf{E}(X|\mathcal{G})\Big) = \mathbf{E}(X) \tag{2.17}$$

which is obtained by taking $A = \Omega$ in (2.13).

Lemma 2.21. *If E is a separable Banach space and X is an E-valued random variable on $(\Omega, \mathcal{F}, P)$ with $\mathbf{E}(\|X\|) < \infty$ and $\mathcal{G} \subset \mathcal{F}$, then*

$$l\big(\mathbf{E}(X|\mathcal{G})\big) = \mathbf{E}(l(X)|\mathcal{G}), \quad \textit{for all } l \in E^*. \tag{2.18}$$

Proof. By definition, the right hand side of (2.18) is $\mathcal{G}$-measurable. The left hand side is $\mathcal{G}$-measurable, too, since l is continuous. For all $A \in \mathcal{G}$,

$$\begin{aligned}\int_A l(\mathbf{E}(X\,|\,\mathcal{G}))\,\mathrm{d}P &= l\Big(\int_A \mathbf{E}(X|\mathcal{G})\,\mathrm{d}P\Big) = l\Big(\int_A X\mathrm{d}P\Big)\\ &= \int_A l(X)\,\mathrm{d}P = \int_A \mathbf{E}(l(X)|\mathcal{G})\,\mathrm{d}P.\end{aligned}$$

By uniqueness of conditional expectation the statement follows. □

Corollary 2.22. *Let E be a separable Banach space, let the random variable $X \in L_1(\Omega, \mathcal{F}, P; E)$, and let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. If X is independent of $\mathcal{G}$, then $l(X)$ is independent of $\mathcal{G}$ for all $l \in E^*$ and $\mathbf{E}(X|\mathcal{G}) = \mathbf{E}(X)$.*

Proof. Let $l \in E^*$, $A \in \mathcal{B}(\mathbf{R})$, and $B \in \mathcal{G}$. Then

$$\begin{aligned}P(\{l(X) \in A\} \cap B) &= P(\{X \in l^{-1}(A)\} \cap B\})\\ &= P(\{X \in l^{-1}(A)\})\,P(B) = P(\{l(X) \in A\})\,P(B),\end{aligned}$$

and hence $l(X)$ is independent of $\mathcal{G}$. Thus, using the corresponding result for real random variables, and Lemma 2.21, we get $l(\mathbf{E}(X|\mathcal{G})) =$

$\mathbf{E}(l(X)|\mathcal{G}) = \mathbf{E}(l(X)) = l(\mathbf{E}(X))$ almost surely. The proof can be completed in the same fashion as in the proof of the uniqueness part of Theorem 2.20, by taking l from a countable subset of E^* separating points of E. □

Definition 2.23. *Let $M(t)_{t\geq 0}$ be an E-valued stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, P)$. The process $\{M(t)\}_{t\geq 0}$ is called a martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ (or $\{M(t)\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-martingale) if*

1. $\mathbf{E}(\|M(t)\|) < \infty$ *for* $t \geq 0$;
2. $\{M(t)\}_{t\geq 0}$ *is adapted to* $\{\mathcal{F}_t\}_{t\geq 0}$;
3. $\mathbf{E}(M(t)|\mathcal{F}_s) = M(s)$ *for* $0 \leq s \leq t < \infty$.

Note that, by (2.17), $\mathbf{E}(M(s)) = \mathbf{E}\big(\mathbf{E}(M(t)|\mathcal{F}_s)\big) = \mathbf{E}(M(t))$ and thus $\mathbf{E}(M(t)) = \mathbf{E}(M(0))$ for $t \geq 0$. This shows that, in the first condition of the definition, it would be enough to assume that $\mathbf{E}(\|M(0)\|) < \infty$. Theorem 2.24 below shows that known theorems about real-valued martingales can be transferred to Banach space-valued martingales by applying functionals.

Theorem 2.24. *Let E be a separable Banach space, let $\{M(t)\}_{t\geq 0}$ be an E-valued process on $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}_t\}_{t\geq 0} \subset \mathcal{F}$ be a filtration. If $\{M(t)\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-martingale, then $\{l(M(t))\}_{t\geq 0}$ is a real-valued $\{\mathcal{F}_t\}_{t\geq 0}$-martingale for all $l \in E^*$. If $E(\|M(t)\|) < \infty$ for all $t \geq 0$, then the converse holds as well.*

Proof. Assume first that $\{M(t)\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-martingale. Then, for all $l \in E^*$,

$$\begin{aligned}\mathbf{E}(|l(M(t))|) &= \int_\Omega |l(M(t))|\,\mathrm{d}P \leq \int_\Omega \|l\|\,\|M(t)\|\,\mathrm{d}P \\ &= \|l\|\mathbf{E}(\|M(t)\|) < \infty.\end{aligned}$$

Since $M(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$ and l is continuous, it follows that $l(M(t))$ is $\mathcal{F}_t$-measurable for all $t \geq 0$ and $l \in E^*$. Finally, by Lemma 2.21, for all $l \in E^*$,

$$\mathbf{E}(l(M(t)|\mathcal{F}_s) = l(\mathbf{E}(M(t)|\mathcal{F}_s)) = l(M(s)), \quad 0 \leq s \leq t.$$

Therefore, $l(M)$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-martingale.

We now prove the converse statement. By assumption, $E(\|M(t)\|) < \infty$. The measurability of $M(t)$ with respect to $\mathcal{F}_t$, for all $t \geq 0$, follows from that of $l(M(t))$ by Corollary 2.24. Since $\{l(M(t))\}_{t\geq 0}$ is a martingale for all $l \in E^*$ it follows that

$$\mathbf{E}(l(M(t))|\mathcal{F}_s) = l(M(s)), \quad 0 \leq s \leq t,$$

which implies, by Lemma 2.21, that

$$l(\mathbf{E}(M(t)|\mathcal{F}_s)) = l(M(s)), \quad 0 \leq s \leq t.$$

The proof can be completed, as in the proof of the uniqueness part of Theorem 2.20, by taking l from a countable subset of E^* separating points of E. □

Remark 2.25. The assumption $\mathbf{E}(\|M(t)\|) < \infty$ in Theorem 2.24 is essential, that is, it is possible that $l(Z) \in L_1(\Omega, \mathcal{F}, P; \mathbf{R})$ for all $l \in E^*$ but $Z \notin L_1(\Omega, \mathcal{F}, P; E)$.

Proof. Let $E := c_0$ be the Banach space of all complex sequences $X = \{X_n\}_{n\in\mathbf{N}}$ with $\lim_{n\to\infty} X_n = 0$, endowed with norm $\|X\| = \sup_{n\in\mathbf{N}} |X_n|$. Let $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}([0,1]), m)$, where m is the Lebesgue measure on $[0,1]$, and let $Z = (Z_n)_{n\in\mathbf{N}}$ be given by $Z_n(\omega) = n1_{(0,\frac{1}{n}]}(\omega)$ for $\omega \in [0,1]$. It is well known that c_0^* can be identified with l_1, the space of complex sequences $l = \{l_n\}$ such that $\sum_{n=1}^{\infty} |l_n| < \infty$, endowed with norm $\|l\| = \sum_{n=1}^{\infty} |l_n|$. It is also known that c_0 is separable. If $l = \{l_n\} \in c_0^*$, then $l(Z) = \sum_{n=1}^{\infty} l_n n 1_{[0,\frac{1}{n}]}$ and $l(Z)$ is thus measurable for all $l \in c_0^*$. Therefore, by Lemma 2.19, Z is measurable. Also,

$$\mathbf{E}\big(|l(Z)|\big) \leq \sum_{n=1}^{\infty} \int_0^1 |l_n| n 1_{(0,\frac{1}{n}]} \, \mathrm{d}m = \sum_{n=1}^{\infty} |l_n| = \|l\|,$$

and thus $l(Z) \in L_1(\Omega, \mathcal{F}, P; \mathbf{R})$ for all $l \in c_0^*$. But $\|Z(\omega)\| = n$ if $\omega \in (\frac{1}{n+1}, \frac{1}{n}]$ and hence

$$\int_0^1 \|Z(\omega)\| \, \mathrm{d}m(\omega) = \sum_{n=1}^{\infty} \int_{(\frac{1}{n+1},\frac{1}{n}]} \|Z(\omega)\| \, \mathrm{d}m(\omega) = \sum_{n=1}^{\infty} n \frac{1}{n(n+1)} = \infty.$$

□

Recall that Jensen's inequality for real-valued conditional expectation states that if $\psi : \mathbf{R} \to \mathbf{R}$ is convex, if the random variables $X, \psi(X) \in L_1(\Omega, \mathcal{F}, P; \mathbf{R})$, and if $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, then

$$\psi(\mathbf{E}(X|\mathcal{G})) \leq \mathbf{E}(\psi(X)|\mathcal{G}). \tag{2.19}$$

Theorem 2.26 (Doob's maximal inequality)**.** *Let E be a separable Banach space and let $\{M(t)\}_{t\geq 0}$ be an E-valued $\{\mathcal{F}_t\}_{t\geq 0}$-martingale. If $M(t) \in L_p(\Omega, \mathcal{F}, P; E)$, $t \geq 0$, for some $p \in [1, \infty)$, then $\{\|M(t)\|^p\}_{t\geq 0}$ is a non-negative real-valued $\{\mathcal{F}_t\}_{t\geq 0}$-submartingale, that is,*

$$\|M(s)\|^p \leq \mathbf{E}\big(\|M(t)\|^p \big| \mathcal{F}_s\big), \quad 0 \leq s \leq t. \tag{2.20}$$

Moreover, if $p > 1$ and $T > 0$, then

$$\mathbf{E}\Big(\sup_{t\in[0,T]} \|M(t\|^p\Big) \le \Big(\frac{p}{p-1}\Big)^p \mathbf{E}(\|M(T)\|^p).$$

Proof. That $\{\|M(t)\|^p\}_{t\ge0}$ is a non-negative real-valued $\{\mathcal{F}_t\}_{t\ge0}$-submartingale follows from Definition 2.23, (2.14), and Jensen's inequality (2.19) with $\psi(x) = x^p$, because, for $0 \le s \le t$,

$$\|M(s)\|^p = \|\mathbf{E}(M(t)|\mathcal{F}_s)\|^p \le \Big(\mathbf{E}\big(\|M(t)\|\,\big|\mathcal{F}_s\big)\Big)^p \le \mathbf{E}\big(\|M(t)\|^p\big|\mathcal{F}_s\big).$$

Now the rest of the proof is a direct consequence of Doob's maximal inequality for positive real-valued submartingales. □

Next we define one of the most important spaces that we will work with when defining the stochastic integral. Let $T > 0$ and define

$$\begin{aligned}\mathcal{M}_T^2(E) := \Big\{ &\{M(t)\}_{t\in[0,T]} : t \mapsto M(t) \text{ is continuous } P\text{-a.s.},\\ &\{M(t)\}_{t\in[0,T]} \text{ is an } E\text{-valued } \{\mathcal{F}_t\}_{t\in[0,T]}\text{-martingale},\\ &\text{and } \sup_{t\in[0,T]} \int_\Omega \|M(t)\|^2 \,\mathrm{d}P < \infty \Big\}\end{aligned}$$

endowed with norm

$$\|M\|_{\mathcal{M}_T^2(E)} := \sup_{t\in[0,T]} \Big(\mathbf{E}(\|M(t)\|^2\Big)^{1/2} = \Big(\mathbf{E}(\|M(T)\|^2\Big)^{1/2}.$$

In the last equality we used $\mathbf{E}(\|M(t)\|)^2 \le \mathbf{E}(\|M(T)\|)^2$, which follows from (2.20) by taking expectations.

Proposition 2.27. *The space $\mathcal{M}_T^2(E)$ is a Banach space and for all $M \in \mathcal{M}_T^2(E)$, we have*

$$\|M\|_{\mathcal{M}_T^2(E)} \le \Big(\mathbf{E}\Big(\sup_{t\in[0,T]} \|M(t)\|^2\Big)\Big)^{1/2} \le 2\|M\|_{\mathcal{M}_T^2(E)}. \tag{2.21}$$

Proof. The first inequality in (2.21) is obvious and the second one follows immediately from Theorem 2.26. Let $\{M_n\} \subset \mathcal{M}_T^2(E)$ be a Cauchy sequence. Then, by (2.21), it is a Cauchy sequence in

$$\mathcal{X} := L_2\big(\Omega, \mathcal{F}, P; C([0,T], E)\big)$$

as well. But $\mathcal{X}$ is a Banach space. Thus $\{M_n\}$ converges to an almost surely continuous process M in $\mathcal{X}$, and also in the norm of $\mathcal{M}_T^2(E)$ in view of (2.21). Finally, to see that M is a martingale, we observe that $M_n(t) \to M$ in $L_2(\Omega, \mathcal{F}, P; E)$, for all $t \in [0,T]$ and hence also in $L_1(\Omega, \mathcal{F}, P; E)$, as $n \to \infty$. Thus, there is a subsequence $\{M_{n_k}\}$ of $\{M_n\}$, which converges to M almost surely, too. If $0 \le s \le t \le T$, then $M_{n_k}(s) = \mathbf{E}(M_{n_k}(t)|\mathcal{F}_s)$ almost surely. By letting $k \to \infty$ the proof is complete. □

We now apply this to a Q-Wiener process on an Hilbert space U as in Proposition 2.16.

Proposition 2.28. *Let $\{W(t)\}_{t\ge 0}$ be a U-valued Q-Wiener process with respect to a normal filtration $\{\mathcal{F}\}_{t\ge 0}$ on $(\Omega, \mathcal{F}, P)$. Then $W \in \mathcal{M}_T^2(U)$ for all $T > 0$.*

Proof. By definition $W \in C([0,T], U)$ almost surely and

$$\mathbf{E}(\|W(t)\|^2) = t\,\mathrm{Tr}(Q) \le T\,\mathrm{Tr}(Q) < \infty, \quad \text{for all } t \in [0,T].$$

Also, by assumption, $\{W(t)\}_{t\ge 0}$ is adapted to $\{\mathcal{F}\}_{t\ge 0}$. Thus, it remains to show that $W(s) = \mathbf{E}(W(t)|\mathcal{F}_s)$ for $0 \le s \le t$ or, equivalently, $\int_A W(s)\,\mathrm{d}P = \int_A W(t)\,\mathrm{d}P$ for $0 \le s \le t$ and $A \in \mathcal{F}_s$. But $W(t) - W(s)$ is independent of $\mathcal{F}_s$ by assumption and thus, by Corollary 2.22,

$$\begin{aligned}\int_A (W(t) - W(s))\,\mathrm{d}P &= \int_A \mathbf{E}(W(t) - W(s)|\mathcal{F}_s)\,\mathrm{d}P \\ &= \int_A \mathrm{d}P\,\mathbf{E}(W(t) - W(s)) = 0.\end{aligned}$$

□

2.4 Measurability of operator valued random variables

We are going to integrate operator-valued processes against a Wiener process. We therefore discuss briefly various concepts of measurability of operator-valued random variables. Since the space of bounded linear operators $L(U,H)$ becomes a Banach space with respect to the operator norm $\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$, $T \in L(U,H)$, it is tempting to endow $L(U,H)$ with its *uniform Borel* σ*-algebra* $\mathcal{B}_{\text{uni}}(L(U,H))$, that is, the smallest σ-algebra which contains all open balls

$$B_r(T) = \{L \in L(U,H) : \|L-T\| < r\}, \quad r > 0,\ T \in L(U,H).$$

This leads to a σ-algebra with too many measurable sets. To see this, we show that, in general, $L(U,H)$ is not separable and thus it has too many open sets. This implies that the class of measurable $L(U,H)$-valued functions is so small that even very simple operator-valued functions are not measurable. Let $U := H := L_2(\mathbf{R})$. We show that $(L(H), \|\cdot\|)$ is not separable. Define the function $S : \mathbf{R} \to L(H)$ by $(S(t)f)(x) = f(x+t)$, $f \in H$. If $t > s$ and $f \in H$, then

$$\|S(t)f - S(s)f\|_H = \|S(s)(S(t-s)f - f)\|_H = \|S(t-s)f - f\|_H.$$

Take $f \in H$ such that $\text{supp}(f) \subset \left(\frac{s-t}{2}, \frac{t-s}{2}\right)$. Then, $\text{supp}(f) \cap \text{supp}(S(t-s)f) = \emptyset$, and thus f and $S(t-s)f$ are orthogonal in H. Therefore,

$$\|S(t-s)f - f\|^2 = \|S(t-s)f\|^2 + \|f\|^2 = 2\|f\|^2,$$

which implies that $\|S(t)f - S(s)f\|^2 = 2\|f\|^2$. Hence, $\|S(t) - S(s)\| \geq \sqrt{2}$ implying that $(L(H), \|\cdot\|)$ is not separable. Next, consider S as a mapping

$$S : (\mathbf{R}, \mathcal{B}(\mathbf{R})) \to (L(H), \mathcal{B}_{\text{uni}}(L(H))).$$

Consider a set $A \notin \mathcal{B}(\mathbf{R})$ and define

$$D := \bigcup_{t \in A} \Big\{ G \in L(H) : \|G - S(t)\| < \frac{\sqrt{2}}{2} \Big\}.$$

This is an open set and hence $D \subset \mathcal{B}_{\text{uni}}(L(H))$. But $S^{-1}(D) = \{s \in \mathbf{R} : S(s) \in D\} = A$. Therefore, S is not measurable.

Instead we consider the *strong Borel* σ*-algebra* of $L(U,H)$ denoted by $\mathcal{B}_{\text{str}}(L(U,H))$, or simply $\mathcal{B}(L(U,H)$, which is defined to be the smallest σ-algebra containing all sets of the form

$$\Big\{ T \in L(U,H) : Tx \in A,\ A \in \mathcal{B}(H),\ x \in U \Big\}.$$

Definition 2.29. *Let $(\Omega, \mathcal{F})$ be a measure space and $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. A mapping $L : (\Omega, \mathcal{F}) \to L(U,H)$ is said to be strongly $\mathcal{G}$-measurable if it is $\mathcal{G}$-measurable if we endow $L(U,H)$ with the strong Borel σ-algebra $\mathcal{B}_{str}(L(U,H))$, that is, if $Lx : (\Omega, \mathcal{F}) \to (H, \mathcal{B}(H))$ is $\mathcal{G}$-measurable for all $x \in U$. If $\mathcal{G} = \mathcal{F}$, then L is said to be strongly measurable.*

One can check that the mapping S considered above is continuous with respect to the strong operator topology of $L(H)$, that is, $t \mapsto S(t)x$ is continuous for every $x \in H$, and it is therefore strongly measurable.

Thus, in general, $\mathcal{B}_{\text{str}}(L(U,H)) \subsetneq \mathcal{B}_{\text{uni}}(L(U,H))$ (strict inclusion). To see that this is not always the case, consider $L(H, \mathbf{R})$, where H is separable Hilbert space. By the Riesz representation theorem $L(H, \mathbf{R})$ and H are isometrically isomorphic and hence $L(H, \mathbf{R})$ is separable. Thus, if we identify $L(H,\mathbf{R})$ and H under the Riesz isomorphism, then by Lemma 2.18,

$$\begin{aligned}\mathcal{B}_{\text{uni}}(L(H,\mathbf{R})) = \mathcal{B}(H) &= \sigma(\{x : l(x) \le \alpha,\ l \in H^*,\ \alpha \in \mathbf{R}\}) \\ &= \sigma(\{x : \langle y,x\rangle \le \alpha,\ y \in H,\ \alpha \in \mathbf{R}\}) \\ &= \mathcal{B}_{\text{str}}(L(H,\mathbf{R})).\end{aligned}$$

Next, we prove that $\mathcal{B}(L_2(U,H)) \subset \mathcal{B}_{\text{str}}(L(U,H))$ which implies, in particular, that $L_2(U,H)$ is a strongly measurable subset of $L(U,H)$.

Lemma 2.30. *The containment $\mathcal{B}(L_2(U,H)) \subset \mathcal{B}_{str}(L(U,H))$ holds.*

Proof. It is enough to show that every open ball in $L_2(U,H)$ also belongs to $\mathcal{B}_{\text{str}}(L(U,H))$. Indeed, if $\{f_k\}$ is an orthonormal basis for U and $T_0 \in L_2(U,H)$, then

$$\begin{aligned}&\Big\{T \in L_2(U,H) : \|T - T_0\|_{L_2} < r\Big\} \\ &= \Big\{T \in L(U,H) : \|T - T_0\|_{L_2} < r\Big\} \\ &= \bigcup_{m=1}^{\infty} \Big\{T \in L(U,H) : \|T - T_0\|_{L_2}^2 \le r^2(1 - \frac{1}{m})\Big\} \\ &= \bigcup_{m=1}^{\infty} \Big\{T \in L(U,H) : \sum_{k=1}^{\infty} \|(T - T_0)f_k\|_H^2 \le r^2(1 - \frac{1}{m})\Big\} \\ &= \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty} \Big\{T \in L(U,H) : \sum_{k=1}^{n} \|(T - T_0)f_k\|_H^2 \le r^2(1 - \frac{1}{m})\Big\}.\end{aligned} \tag{2.22}$$

The map

$$L(U,H) \to H^n := \overbrace{H \times \cdots \times H}^{n} \to \mathbf{R}$$
$$T \mapsto \big((T - T_0)f_1, \cdots, (T - T_0)f_n\big) \mapsto \sum_{k=1}^{n} \|(T - T_0)f_k\|^2$$

is continuous if we endow $L(U,H)$ with the strong topology, H^n with the product topology and $\mathbf{R}$ with the natural topology. Thus it is measurable with respect to the Borel σ-algebras generated by the respective topologies. Thus, the set in (2.22) belongs to $\mathcal{B}_{\mathrm{Str}}(L(U,H))$ and the proof is complete. □

Finally, we will need the following lemma.

Lemma 2.31. *Let L be and $L(U,H)$-valued strongly measurable mapping and ξ be a U-valued measurable mapping on a measurable space $(\Omega,\mathcal{F})$. Then $L\xi$ is an H-valued measurable mapping on $(\Omega,\mathcal{F})$.*

Proof. Since H is separable, $L\xi$ is measurable if and only if $\langle L\xi,x\rangle$ is $\mathbf{R}$-valued measurable for all $x\in H$ by Lemma 2.18. Let $\{e_k\}$ be an orthonormal basis for U. Then, $\langle L\xi,x\rangle = \langle \xi,L^*x\rangle = \sum_k \langle \xi,e_k\rangle\langle x,Le_k\rangle$ is measurable as both ξ and Le_k are measurable and hence weakly measurable and the sum converges for all $\omega\in\Omega$. □

3 The stochastic integral for nuclear Wiener processes

Let $(U,\langle\cdot,\cdot\rangle_U)$ and $(H,\langle\cdot,\cdot\rangle_H)$ be separable Hilbert spaces and assume that $\{W(t)\}_{t\in[0,T]}$ is a U-valued Q-Wiener process on a probability space $(\Omega,\mathcal{F},P)$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, where $T>0$ is fixed. Let $\Omega_T=[0,T]\times\Omega$ and $P_T=m\times P$, where m is the Lebesgue measure on $[0,T]$, be the product measure on Ω_T. We first define the stochastic integral for elementary processes.

3.1 The stochastic integral for elementary processes

Definition 3.1. *An $L(U,H)$-valued process $\{\Phi(t)\}_{t\in[0,T]}$ is called elementary if there exist $0 = t_0 < t_1 < \cdots < t_N = T$, $N \in \mathbf{N}$, such that*

$$\Phi(t) = \sum_{m=0}^{N-1} \Phi_m 1_{(t_m,t_{m+1}]}(t), \quad t \in [0,T],$$

where

- $\Phi_m : (\Omega, \mathcal{F}) \to L(U,H)$ *is strongly $\mathcal{F}_{t_m}$-measurable;*
- Φ_m *takes only a finite number of values in $L(U,H)$, that is,*

$$\Phi_m(\omega) = \sum_{j=1}^{k_m} 1_{\Omega_j^m}(\omega) L_j^m,$$

where $L_j^m \in L(U,H)$ and $\Omega = \bigcup_{j=1}^{k_m} \Omega_j^m$ with the union being disjoint.

We denote the (linear) space of elementary process by $\mathcal{E}$.

For $\Phi \in \mathcal{E}$, define

$$\text{(3.1)} \qquad \operatorname{Int}(\Phi)(t) = \int_0^t \Phi \, \mathrm{d}W := \sum_{n=0}^{N-1} \Phi_n(\Delta W_n(t)), \quad t \in [0,T],$$

where

$$\Delta W_n(t) = W(t_{n+1} \wedge t) - W(t_n \wedge t),$$

and $t \wedge s = \min(t,s)$. Note that $\Delta W_n(0) = 0$ and that for $t \in (t_k, t_{k+1}]$ we have

$$\Delta W_n(t) = \begin{cases} W(t_{n+1}) - W(t_n), & t_n < t_k, \\ W(t) - W(t_k), & t_n = t_k, \\ 0, & t_n > t_k. \end{cases}$$

Note also that $\Delta W_n(t)$ is $\mathcal{F}_t$-measurable and that $\Delta W_n(t)$ is independent of $\mathcal{F}_s$ for $s \leq t_n$. We recall the following result for real-valued random variables.

Lemma 3.2. *Let X and Y be real-valued random variables on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. If X is $\mathcal{G}$-measurable and $Y, XY \in L_1(\Omega, \mathcal{F}, P; \mathbf{R})$, then $\mathbf{E}(XY|\mathcal{G}) = X\,\mathbf{E}(Y|\mathcal{G})$.*

Proof. The proof is elementary and is left to the reader. □

Proposition 3.3. *For all $\Phi \in \mathcal{E}$, the integral $\{\mathrm{Int}(\Phi)(t)\}_{t\in[0,T]}$ defined in* (3.1) *is a continuous square integrable $\{\mathcal{F}_t\}_{t\in[0,T]}$-martingale, that is, $\{\mathrm{Int}(\Phi)(t)\}_{t\in[0,T]} \in \mathcal{M}_T^2(H)$.*

Proof. Let $M(t) := \int_0^t \Phi\,\mathrm{d}W$, $t \in [0,T]$. Then, $M : [0,T] \to H$ is continuous a.s., because $\Delta W_n(t) : [0,T] \to U$ is continuous a.s. and $\Phi : U \to H$ is continuous for all $\omega \in \Omega$. The process $\{M(t)\}_{t\in[0,T]}$ is square integrable because

$$\begin{aligned}
\mathbf{E}(\|M(t)\|^2) &= \mathbf{E}\Big(\Big\|\sum_{n=0}^{N-1} \Phi_n(\Delta W_n(t))\Big\|^2\Big) \\
&\leq \mathbf{E}\Big(\sum_{n=0}^{N-1} \|\Phi_n\|_{L(U,H)}^2 \|\Delta W_n(t)\|_U^2\Big) \\
&\leq \max_n \Big(\sum_{j=1}^{k_n} \|L_j^n\|_{L(U,H)}^2\Big) \sum_{n=0}^{N-1} \underbrace{\mathbf{E}\big(\|\Delta W_n(t)\|^2\big)}_{<\infty}, \; t \in [0,T].
\end{aligned}$$

Finally, we show that M is an $\{\mathcal{F}_t\}_{t\in[0,T]}$-martingale. Clearly, M is integrable because $\mathbf{E}(\|M(t)\|) \leq \mathbf{E}(\|M(t)\|^2) < \infty$, $t \in [0,T]$. Each term $\Phi_n(\Delta W_n(t))$ in $M(t)$, and hence also $M(t)$, is $\mathcal{F}_t$-measurable in view of Lemma 2.31. To prove the martingale property, that is,

$$\int_0^s \Phi\,\mathrm{d}W = \mathbf{E}\Big(\int_0^t \Phi\,\mathrm{d}W \Big| \mathcal{F}_s\Big), \quad s \leq t,$$

let $t \in (t_k, t_{k+1}]$, $s \in (t_l, t_{l+1}]$, and $s \le t$, $l \le k$. Then

$$\begin{aligned}
\int_0^t \Phi \, \mathrm{d}W &= \sum_{n=0}^{N-1} \Phi_n(\Delta W_n(t)) - \Phi_l(W(s)) + \Phi_l(W(s)) \\
&= \sum_{n=0}^{l-1} \Phi_m(\Delta W_n(t)) + \Phi_l((W(s) - W(t_l)) \\
&\quad + \Phi_l(W(t_{l+1} \wedge t) - W(s)) + \sum_{n=l+1}^{N-1} \Phi_n(\Delta W_n(t)) \\
&= \sum_{n=0}^{l} \Phi_n(\Delta W_n(s)) + \Phi_l(W(t_{l+1} \wedge t) - W(s)) + \sum_{n=l+1}^{N-1} \Phi_n(\Delta W_n(t)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{E}\Big(\int_0^t \Phi \, \mathrm{d}W \Big| \mathcal{F}_s\Big) &= \mathbf{E}\Big(\int_0^s \Phi \, \mathrm{d}W \Big| \mathcal{F}_s\Big) \\
&\quad + \mathbf{E}\big(\Phi_l(W(t_{l+1} \wedge t) - W(s)) \big| \mathcal{F}_s\big) + \mathbf{E}\Big(\sum_{l+1}^{N-1} \Phi_n(\Delta W_n(t)) \Big| \mathcal{F}_s\Big).
\end{aligned}$$

We saw that $\int_0^s \Phi \, \mathrm{d}W$ is $\mathcal{F}_s$-measurable and hence $\mathbf{E}(\int_0^s \Phi \, \mathrm{d}W | \mathcal{F}_s) = \int_0^s \Phi \, \mathrm{d}W$. For the second term, let $\{e_k\}$ be an orthonormal basis of U. Then, for all $x \in H$, using that Φ_l is strongly (hence weakly) $\mathcal{F}_s$-measurable and that $W(t_{l+1} \wedge s)$ is independent of $\mathcal{F}_s$, by Lemma 3.2,

$$\begin{aligned}
&\Big\langle \mathbf{E}\Big(\Phi_l(W(t_{l+1} \wedge t) - W(s)) \Big| \mathcal{F}_s\Big), x \Big\rangle \\
&\quad = \sum_k \mathbf{E}\Big(\langle W(t_{l+1} \wedge t) - W(s), e_k \rangle \langle \Phi_l e_k, x \rangle \Big| \mathcal{F}_s\Big) \\
&\quad = \sum_k \langle \Phi_l e_k, x \rangle \mathbf{E}\Big(\langle W(t_{l+1} \wedge t) - W(s), e_k \rangle \Big| \mathcal{F}_s\Big) \\
&\quad = \sum_k \langle \Phi_l e_k, x \rangle \mathbf{E}\big(\langle W(t_{l+1} \wedge t) - W(s), e_k \rangle\big) = 0.
\end{aligned}$$

This shows that $\mathbf{E}(\Phi_l(W(t_{l+1} \wedge t) - W(s)) | \mathcal{F}_s) = 0$. The rest of the terms are of the form $\mathbf{E}(\Phi_m(W(\tau) - W(\sigma) | \mathcal{F}_s))$, where $s \le \sigma \le \tau \le t$, $t_m \le \sigma$.

Let $A \in \mathcal{F}_s$. Then

$$\int_A \Phi_m(W(\tau)-W(\sigma))\,\mathrm{d}P = \int_A \sum_j 1_{\Omega_j^m} L_j^m \big(W(\tau)-W(\sigma)\big)\,\mathrm{d}P$$
$$= \sum_j L_j^n \int_{A\cap\Omega_j^m} (W(\tau)-W(\sigma))\,\mathrm{d}P.$$

Since $A \in \mathcal{F}_s \subset \mathcal{F}_\sigma$ and $\Omega_j^n \in \mathcal{F}_{t_m} \subset \mathcal{F}_\sigma$ and $W(\tau)-W(\sigma)$ is independent of $\mathcal{F}_\sigma$, it follows from Corollary 2.22 and the definition of the conditional expectation that

$$\sum_j L_j^n \int_{A\cap\Omega_j^m} (W(\tau)-W(\sigma))\,\mathrm{d}P = \sum_j L_j^n \int_{A\cap\Omega_j^m} \mathbf{E}(W(\tau)-W(\sigma)\,|\,\mathcal{F}_\sigma)\,\mathrm{d}P$$
$$= \sum_j L_j^n \int_{A\cap\Omega_j^m} \mathbf{E}(W(\tau)-W(\sigma))\,\mathrm{d}P = 0.$$

Thus, by the uniqueness of the conditional expectation, it follows that $\mathbf{E}(\Phi_m(W(\tau)-W(\sigma)|\mathcal{F}_s)) = 0$. □

Remark 3.4. Since $M = \int \Phi\,\mathrm{d}W$ is a martingale it follows that

$$\mathbf{E}\Big(\int_0^t \Phi\,\mathrm{d}W\Big) = \mathbf{E}(M(0)) = 0.$$

For $\Phi \in \mathcal{E}$, define

$$\|\Phi\|_T := \Big(\mathbf{E}\Big(\int_0^T \Big\|\Phi(s)Q^{1/2}\Big\|^2_{L_2(U,H)}\,\mathrm{d}s\Big)\Big)^{1/2}$$

or, equivalently,

$$\|\Phi\|_T^2 = \mathbf{E}\Big(\int_0^T \|\Phi(s)\|^2_{L_2^0}\,\mathrm{d}s\Big).$$

The following identity, called the Itô-isometry, will be crucial when we extend the stochastic integral to a larger class of integrands.

Proposition 3.5 (Itô-isometry)**.** *If* $\Phi \in \mathcal{E}$*, then*

$$\mathbf{E}\Big(\Big\|\int_0^T \Phi\,\mathrm{d}W\Big\|^2\Big) = \mathbf{E}\Big(\int_0^T \Big\|\Phi(s)Q^{1/2}\Big\|^2_{L_2(U,H)}\,\mathrm{d}s\Big), \tag{3.2}$$

or, equivalently,

$$\Big\|\int_0^{\cdot} \Phi\,\mathrm{d}W\Big\|_{\mathcal{M}_T^2(H)} = \|\Phi\|_T.$$

Proof. Let $\Phi \in \mathcal{E}$. By definition,

$$\int_0^T \Phi\,\mathrm{d}W = \sum_{n=0}^{N-1} \Phi_n(\Delta W_n),$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$. Then,

$$\begin{aligned}
\mathbf{E}\Big(\Big\|\int_0^T \Phi\,\mathrm{d}W\Big\|^2\Big) &= \mathbf{E}\Big(\Big\langle \sum_{n=0}^{N-1} \Phi_n\Delta W_n, \sum_{m=0}^{N-1} \Phi_m\Delta W_m\Big\rangle\Big)\\
&= \mathbf{E}\Big(\sum_{n=0}^{N-1} \|\Phi_n\Delta W_n\|^2\Big) + 2\mathbf{E}\Big(\sum_{m<n} \langle \Phi_n\Delta W_n, \Phi_m\Delta W_m\rangle\Big)\\
&= T_1 + T_2.
\end{aligned}$$

We will show that $T_1 = \|\Phi\|_T^2$ and that $T_2 = 0$. Let $\{f_k\}_{k\in\mathbf{N}}$ be an orthonormal basis of H and $\{e_k\}_{k\in\mathbf{N}}$ be an orthonormal basis of U. Then, using Parseval's formula, the Monotone Convergence Theorem, and the law of double expectation,

$$\begin{aligned}
\mathbf{E}\Big(\|\Phi_n\Delta W_n\|^2\Big) &= \mathbf{E}\Big(\sum_l \langle \Phi_n\Delta W_n, f_l\rangle^2\Big) = \sum_l \mathbf{E}\Big(\langle \Phi_n\Delta W_n, f_l\rangle^2\Big)\\
&= \sum_l \mathbf{E}\Big(\mathbf{E}\big(\langle \Phi_n\Delta W_n, f_l\rangle^2 \big| \mathcal{F}_{t_n}\big)\Big) = \sum_l \mathbf{E}\Big(\mathbf{E}\big(\langle \Delta W_n, \Phi_n^* f_l\rangle^2 \big| \mathcal{F}_{t_n}\big)\Big).
\end{aligned}$$

By Parseval's formula in U,

$$\begin{aligned}
\langle \Delta W_n, \Phi_n^* f_l\rangle^2 &= \Big(\sum_k \langle \Delta W_n, e_k\rangle \langle \Phi_n^* f_l, e_k\rangle\Big)^2\\
&= \Big(\sum_k \underbrace{\langle f_l, \Phi_n e_k\rangle}_{=a_k} \underbrace{\langle \Delta W_n, e_k\rangle}_{=b_k}\Big)^2 = \Big(\sum_k a_k b_k\Big)^2 = \sum_{k,j} a_k a_j b_k b_j.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{E}\Big(\langle \Delta W_n, \Phi_n^* f_l\rangle^2 \Big| \mathcal{F}_{t_n}\Big) &= \mathbf{E}\Big(\sum_{k,j} a_k a_j b_k b_j \Big| \mathcal{F}_{t_n}\Big) \\
&= \sum_{k,j} \mathbf{E}\Big(a_k a_j b_k b_j \Big| \mathcal{F}_{t_n}\Big) \qquad (3.3) \\
&= \sum_{k,j} a_k a_j \mathbf{E}\big(b_k b_j \big| \mathcal{F}_{t_n}\big) = \sum_{k,j} a_k a_j \mathbf{E}(b_k b_j) \qquad (3.4) \\
&= \sum_{k,j} \langle f_l, \Phi_n e_k\rangle \langle f_l, \Phi_n e_j\rangle \mathbf{E}(\langle \Delta W_n, e_k\rangle \langle \Delta W_n, e_j\rangle) \\
&= \sum_{k,j} \langle f_l, \Phi_n e_k\rangle \langle f_l, \Phi_n e_j\rangle \Delta t_n \langle Q e_k, e_j\rangle \qquad (3.5) \\
&= \Delta t_n \sum_k \Big\langle Q\langle \Phi_n^* f_l, e_k\rangle e_k, \sum_j \langle \Phi_n^* f_l, e_j\rangle e_j \Big\rangle \\
&= \Delta t_n \sum_k \langle Q\langle \Phi_n^* f_l, e_k\rangle e_k, \Phi_n^* f_l\rangle \\
&= \Delta t_n \Big\langle \sum_k \langle \Phi_n^* f_l, e_k\rangle e_k, Q\Phi_n^* f_l \Big\rangle \\
&= \Delta t_n \|Q^{1/2}\Phi_n^* f_l\|^2 \quad a.s.
\end{aligned}$$

We used the Dominated Convergence Theorem in (3.3), Corollary 2.22 on **R** and Lemma 3.2 in (3.4), and the assumption on the increments of a Q-Wiener process in (3.5). Hence, using property (1) in Remark 1.4,

$$\begin{aligned}
T_1 &= \mathbf{E}\Big(\sum_l \sum_{n=0}^{N-1} \Delta t_n \|Q^{1/2}\Phi_n^* f_l\|^2\Big) = \mathbf{E}\Big(\sum_{n=0}^{N-1} \Delta t_n \|Q^{1/2}\Phi_n^*\|^2_{L_2(H,U)}\Big) \\
&= \mathbf{E}\Big(\sum_{n=0}^{N-1} \Delta t_n \|\Phi_n Q^{1/2}\|_{L_2(U,H)}\Big) = \mathbf{E}\Big(\int_0^T \|\Phi(s) Q^{1/2}\|_{L_2(U,H)}\, \mathrm{d}s\Big).
\end{aligned}$$

Similarly to the diagonal terms above, for a typical term in T_2, using

Parseval's formula twice, we obtain

$$
\begin{aligned}
&\langle \Phi_n \Delta W_n, \Phi_m \Delta W_m \rangle = \cdots \\
(3.6) \qquad &\quad = \sum_{l,j,k} \langle \Delta W_n, e_k \rangle \langle \Phi_n^* f_l, e_k \rangle \langle \Delta W_m, e_j \rangle \langle \Phi_m^* f_l, e_j \rangle.
\end{aligned}
$$

Finally, using Corollary 2.22 on $\mathbf{R}$ and Lemma 3.2, the expectation of each term in (3.6) equals zero as

$$
\begin{aligned}
&\mathbf{E}\Big(\langle \Delta W_n, e_k \rangle \langle \Phi_n^* f_l, e_k \rangle \langle \Delta W_m, e_j \rangle \langle \Phi_m^* f_l, e_j \rangle \Big) \\
&\quad = \mathbf{E}\Big(\mathbf{E}\big(\langle \Delta W_n, e_k \rangle \langle \Phi_n^* f_l, e_k \rangle \langle \Delta W_m, e_j \rangle \langle \Phi_m^* f_l, e_j \rangle \,\big|\, \mathcal{F}_{t_n} \big) \Big) \\
&\quad = \mathbf{E}\Big(\langle \Phi_n^* f_l, e_k \rangle \langle \Phi_m^* f_l, e_j \rangle \langle \Delta W_m, e_j \rangle \mathbf{E}\big(\langle \Delta W_n, e_k \rangle \,\big|\, \mathcal{F}_{t_n} \big) \Big) = 0.
\end{aligned}
$$

By the Dominated Convergence Theorem we conclude $T_2 = 0$. □

Corollary 3.6. *If* $\Phi_1, \Phi_2 \in \mathcal{E}$, *then*

$$
\begin{aligned}
&\mathbf{E}\Big(\Big\langle \int_0^T \Phi_1 \,\mathrm{d}W, \int_0^T \Phi_2 \,\mathrm{d}W \Big\rangle_H \Big) \\
&\qquad\qquad = \mathbf{E}\Big(\int_0^T \langle \Phi_1(s) Q^{1/2}, \Phi_2(s) Q^{1/2} \rangle_{L_2(U,H)} \,\mathrm{d}s \Big).
\end{aligned}
$$

Proof. The statement follows from Itô's Isometry, the linearity of the integral, and polarization first in H, then in $L_2(U,H)$. □

Remark 3.7. The functional $\|\cdot\|_T$ is only a seminorm on $\mathcal{E}$. Let $\Phi \in \mathcal{E}$ and assume that

$$
\begin{aligned}
&\|\Phi\|_T^2 = \mathbf{E}\Big(\int_0^T \|\Phi(s) Q^{1/2}\|_{L_2(U,H)}^2 \,\mathrm{d}s \Big) \\
&\qquad\qquad = \sum_{k=1}^\infty \mathbf{E}\Big(\int_0^T \|\Phi(s) Q^{1/2} e_k\|^2 \,\mathrm{d}s \Big) = 0.
\end{aligned}
$$

Then $\|\Phi(s)Q^{1/2}e_k\| = 0$, P_T-a.s., for all $k \in \mathbf{N}$, and thus $\Phi(s)Q^{1/2}e_k = 0$, P_T-a.s., for all $k \in \mathbf{N}$, which implies, by countable additivity, that $\Phi(s)Q^{1/2} = 0$, P_T-a.s. Therefore, $\Phi = 0$ on $Q^{1/2}(U)$, P_T-a.s. Let

$$\mathcal{E}_0 := \Big\{\Phi \in \mathcal{E} : \Phi = 0 \text{ on } Q^{1/2}(U),\ P_T\text{-a.s.}\Big\}.$$

We re-define $\mathcal{E}$ to be the quotient space $\mathcal{E} := \mathcal{E}/\mathcal{E}_0$. Then $\|\cdot\|_T$ is a norm on $\mathcal{E}$.

3.2 Extension of the stochastic integral to more general processes

Propositions 3.3 and 3.5 show that the map

$$\mathrm{Int} : \big(\mathcal{E}, \|\cdot\|_T\big) \to \big(\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2}\big)$$

is isometric (hence continuous). Since, by Proposition 2.27, the space $\big(\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2}\big)$ is complete, Int extends uniquely to an isometric mapping to the abstract completion $\overline{\mathcal{E}}$ of $\mathcal{E}$, by

$$\mathrm{Int}(\Phi) := \lim_{n\to\infty} \mathrm{Int}(\Phi_n), \quad \Phi \in \overline{\mathcal{E}}, \quad \{\Phi_n\} \subset \mathcal{E} \text{ with } \lim_{n\to\infty} \Phi_n = \Phi.$$

Since the abstract completion of a normed space contains objects that are hard to work with (equivalence classes), we will characterize $\overline{\mathcal{E}}$ in a different way. Let us introduce the σ-algebras

$$\mathcal{P}_\infty = \sigma\Big(\Big\{(s,t] \times F : 0 \le s < t,\ F \in \mathcal{F}_s\Big\} \cup \Big\{\{0\} \times F : F \in \mathcal{F}_0\Big\}\Big)$$

and

$$\mathcal{P}_T = \sigma\Big(\Big\{(s,t] \times F : 0 \le s < t \le T,\ F \in \mathcal{F}_s\Big\} \cup \Big\{\{0\} \times F : F \in \mathcal{F}_0\Big\}\Big).$$

Definition 3.8. *If $\tilde{H}$ is a separable Hilbert space and $Y : (\Omega_T, \mathcal{P}_T) \to (\tilde{H}, \mathcal{B}(\tilde{H}))$ is measurable, then Y is called $\tilde{H}$-predictable.*

The next proposition shows that the class of predictable processes is rich.

Proposition 3.9. *If H is a separable Hilbert space, then the following σ-algebras coincide.*

1. $\mathcal{P}_1 = \sigma($*adapted continuous processes*$)$
2. $\mathcal{P}_2 = \sigma($*adapted left continuous processes with right hand limits*$)$
3. $\mathcal{P}_3 = \sigma($*adapted left continuous processes*$)$
4. $\mathcal{P}_\infty$

Here, processes are considered as mappings $\Omega \times [0,\infty) \to H$.

Proof. Since H is separable, it is enough to consider $\mathbf{R}$-valued processes by Corollary 2.19. Clearly, $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3$. To show that $\mathcal{P}_3 \subset \mathcal{P}_\infty$, let X be an adapted left continuous process and define

$$X_n(t) = X(0)1_0(t) + \sum_{k=0}^{\infty} X\left(\frac{k}{2^n}\right) 1_{(\frac{k}{2^n}, \frac{k+1}{2^n}]}(t).$$

Then, X_n is an adapted piecewise constant process. Since X is left continuous, it follows that $X_n(t,\omega) \to X(t,\omega)$. But X_n is $\mathcal{P}_\infty$-measurable and therefore X is $\mathcal{P}_\infty$-measurable. Thus, $\mathcal{P}_3 \subset \mathcal{P}_\infty$. Finally, to see that $\mathcal{P}_\infty \subset \mathcal{P}_1$, fix $0 \le s < t$, and let $R = (s,t] \times F$, $F \in \mathcal{F}_s$. Let $\{f_n\}$ be a sequence of trapezoidal functions such that

$$\lim_{n\to\infty} f_n = 1_{(s,t]}, \quad f_n(x) = 0, \quad \text{if } x \in [0,s).$$

Define $X_n := f_n \cdot 1_F$. Then X_n is adapted and continuous and thus $\mathcal{P}_1$-measurable. But $\lim_{n\to\infty} X_n = 1_R$ and this implies that $R \in \mathcal{P}_1$. Let $F \in \mathcal{F}_0$ and define

$$f_n(x) = \begin{cases} 1, & x = 0, \\ -nx+1, & x \in (0, \frac{1}{n}), \\ 0, & x \in [\frac{1}{n}, \infty). \end{cases}$$

Then $X_n := f_n \cdot 1_F$ is adapted and continuous and hence $\mathcal{P}_1$-measurable. But $\lim X_n = 1_{\{0\}\times F}$ and thus $\{0\} \times F \in \mathcal{P}_1$. Therefore, $\mathcal{P}_\infty \subset \mathcal{P}_1$. □

Remark 3.10. Of course, an analogous statement holds for $\mathcal{P}_T$.

Theorem 3.11. *There is an explicit characterization of $\overline{\mathcal{E}}$ given by*

$$\begin{aligned} \mathcal{N}_W^2 &= \mathcal{N}_W^2(0,T;H) \\ &= \Big\{\Phi : [0,T] \times \Omega \to L_2^0 : \Phi \textit{ is } L_2^0\textit{-predictable and } \|\Phi\|_T < \infty\Big\} \\ &= L_2\big([0,T] \times \Omega, \mathcal{P}_T, m \times P; L_2^0\big). \end{aligned}$$

Proof. Since L_2^0 is complete by Lemma 1.5, it follows that

$$L_2(\Omega_T, \mathcal{P}_T, P_T; L_2^0)$$

is complete. By Lemma 1.7, $L(U,H)_0 \subset L_2^0$ and therefore $\Phi \in \mathcal{E}$ is L_2^0-predictable by construction. Thus, we need show that $\mathcal{E}$ is dense in $\mathcal{N}_W^2(0,T;H)$, that is, if Φ is an L_2^0-predictable process such that $\|\Phi\|_T < \infty$, then there is a sequence $\{\Phi_n\} \subset \mathcal{E}$ such that $\|\Phi - \Phi_n\|_T \to 0$ as $n \to \infty$. If $\Phi \in \mathcal{N}_W^2$, then there is a sequence of simple random variables

$$\Phi_n = \sum_{k=1}^{M_n} L_k^n 1_{A_k^n}, \quad A_k \in \mathcal{P}_T,\ L_k^n \in L_2^0,$$

such that $\|\Phi - \Phi_n\|_T \to 0$ (this follows from the construction of the Bochner integral.) Therefore it is enough to consider

$$\Phi = L1_A, \quad L \in L_2^0, A \in \mathcal{P}_T.$$

Let $A \in \mathcal{P}_T$ and $L \in L_2^0$. By Lemma 1.7 there is $\{L_n\} \subset L(U,H)_0$ such that $L_n \to L$ in L_2^0 and by the Dominated Convergence Theorem

$$\|L1_A - L_n 1_A\|_T \to 0 \quad \text{as } n \to \infty.$$

Therefore, we may consider

$$\Phi = L1_A, \quad A \in \mathcal{P}_T, L \in L(U,H)_0.$$

If $\Phi = L1_A$, $A \in \mathcal{P}_T$, $L \in L(U,H)_0$, then we need to show that there is $\{\Phi_n\} \subset \mathcal{E}$ such that $\|\Phi_n - L1_A\|_T \to 0$ as $n \to \infty$. The case $L = 0$ is clear so assume that $L \neq 0$. Let

$$\mathcal{A} := \Big\{(s,t] \times F : 0 \leq s \leq t \leq T,\ F \in \mathcal{F}_s\Big\} \cup \Big\{\{0\} \times F : F \in \mathcal{F}_0)\Big\}$$

be the set of predictable rectangles. Define

$$\begin{aligned} \mathcal{G} := \Big\{A \in \mathcal{P}_T : \ &\text{for all } \varepsilon > 0, \text{ there is } \Lambda = \bigcup_{n=1}^{N} A_n, A_i \in \mathcal{A}, \\ &\text{with } A_i \cap A_j = \emptyset \text{ for } i \neq j \text{ and } P_T((A \setminus \Lambda) \cup (\Lambda \setminus A)) < \varepsilon\Big\} \end{aligned} \tag{3.7}$$

and

$$\mathcal{K} := \Big\{\bigcup_{i \in I} A_i : I \text{ is finite}, A_i \in \mathcal{A}\Big\}.$$

It is not difficult to check that $\mathcal{K}$ is Π-system and $\mathcal{G}$ is a λ-system. Note that $\mathcal{K} \subset \mathcal{G}$ (by writing $A \in \mathcal{K}$ as a disjoint union). By Dynkin's lemma we have $\sigma(\mathcal{K}) \subset \mathcal{G}$. But $\sigma(\mathcal{K}) = \mathcal{P}_T$ and $\mathcal{G} \subset \mathcal{P}_T$ and therefore $\mathcal{P}_T = \mathcal{G}$. Let $A \in \mathcal{P}_T$, $\varepsilon > 0$ and choose Λ as in (3.7) with

$$P_T\big((A \setminus \Lambda) \cup (\Lambda \setminus A)\big) < \frac{\varepsilon}{\|L\|_{L_2^0}}.$$

Then

$$\Big\| L 1_A - \sum_{n=1}^{N} L 1_{A_n} \Big\|_T^2 = \mathbf{E} \int_0^T \Big\| L(1_A - \sum_{n=1}^{N} 1_{A_n}) \Big\|_{L_2^0}^2 \mathrm{d}s \le \|L\|_{L_2^0} \frac{\varepsilon}{\|L\|_{L_2^0}} = \varepsilon.$$

Finally, $\sum_{n=1}^{N} L 1_{A_n}$ only differs (possibly) from an elementary process in a term $L 1_{\{0\}\times F}$, $F \in \mathcal{F}$. But $\|L 1_{\{0\}\times F}\|_T = 0$, so by taking $\Phi = \sum_{n=1}^{N} L 1_{A_n} - L 1_{\{0\}\times F}$, we obtain an elementary process. □

Remark 3.12. Both Itô's Isometry (3.2) and Corollary 3.6 still hold for $\Phi \in \mathcal{N}_W^2$.

Remark 3.13. By a so-called localization procedure, one can extend the class of integrands even further to

$$\mathcal{N}_W = \Big\{ \Phi : \Omega_T \to L_2^0 : \Phi \text{ is predictable}, P\Big(\int_0^T \|\Phi\|_{L_2^0}^2 \mathrm{d}s < \infty \Big) = 1 \Big\}.$$

The integral in this case becomes a local martingale, only, and the Itô isometry does not hold. For the type of equations we study here, this extension is not necessary and therefore we do not pursue this issue any further.

4 Stochastic integral for cylindrical Wiener processes

We would like to consider a Wiener process $\{W(t)\}_{t\ge 0}$ with covariance operator Q such that $\mathrm{Tr}(Q) = \infty$, for example, $Q = I$. Recall that, if

$Q \in L(U)$, $Q \geq 0$, $\mathrm{Tr}(Q) < \infty$, then

$$W(t) = \sum_{k=1}^{\infty} e_k \beta_k(t)$$

where $e_k = \lambda_k^{1/2} f_k$ is an orthonormal basis for $U_0 = Q^{1/2}(U)$ and the series converges in $L_2(\Omega, \mathcal{F}, P; U)$. Note that the inclusion

$$J : (U_0, \langle \cdot, \cdot \rangle_0) \to (U, \langle \cdot, \cdot \rangle), \quad \text{with } x \mapsto Jx = x,$$

is Hilbert-Schmidt if and only if $\mathrm{Tr}(Q) < \infty$. Indeed, if $\{e_k\}$ is an ortonormal basis for U_0, then

$$\begin{aligned} \|J\|^2_{L_2(U_0,U)} &= \sum_k \langle Je_k, Je_k \rangle_U = \sum_k \langle e_k, e_k \rangle_U \\ &= \sum_k \langle Q^{1/2} Q^{-1/2} e_k, Q^{1/2} Q^{-1/2} e_k \rangle_U \\ &= \sum_k \langle Q^{1/2} f_k, Q^{1/2} f_k \rangle_U = \mathrm{Tr}(Q) = \|W(1)\|^2_{L_2(\Omega,U)}, \end{aligned}$$

since $\{f_k\} = \{Q^{-1/2} e_k\}$ is an orthonormal basis for $\left(\ker Q^{1/2}\right)^{\perp}$. Thus, the series defining W converges or diverges in $L_2(\Omega, H)$ depending on whether J is a Hilbert-Schmidt operator or not. Therefore, if $\mathrm{Tr}(Q) = \infty$, then we need to consider another Hilbert space $(\tilde{U}, [\cdot, \cdot])$ with norm $[\![\cdot]\!]$ such that there is an embedding $J : U_0 \to \tilde{U}$ which is Hilbert-Schmidt in order to define a Q-Wiener process.

Remark 4.1. Given $Q \in L(U)$, $Q \geq 0$, we may always find $\tilde{U}$ such that there is an embedding $J : U_0 \to \tilde{U}$. Set $\tilde{U} := U$ and let $\alpha_k > 0$, $k \in \mathbf{N}$, with $\sum_k \alpha_k^2 < \infty$. Define

$$J : U_0 \to U, \quad u \mapsto \sum_k \alpha_k \langle u, e_k \rangle_0 e_k,$$

where $\{e_k\}$ is an orthonormal basis of U_0. Then J is one-to-one and Hilbert-Schmidt. Indeed, if $u, v \in U_0$, then

$$\begin{aligned} u = v &\Leftrightarrow \langle u, e_k\rangle_0 = \langle v, e_k\rangle_0,\ \forall k \in \mathbf{N}, \\ &\Leftrightarrow \alpha_k\langle u, e_k\rangle_0 = \alpha_k\langle v, e_k\rangle_0,\ \forall k \in \mathbf{N}, \\ &\Leftrightarrow \sum_k \alpha_k\langle u, e_k\rangle_0 e_k = \sum_k \alpha_k\langle v, e_k\rangle_0 e_k,\ \forall k \in \mathbf{N}, \\ &\Leftrightarrow J(u) = J(v). \end{aligned}$$

and, since $\|Q^{-1/2}e_k\|_U = \|e_k\|_0$=1,

$$\begin{aligned} \|J\|_{L_2(U_0,\tilde{U})} &= \sum_{k=1}^{\infty} \|Je_k\|_U^2 = \sum_{k=1}^{\infty} \Big\| \sum_{n=1}^{\infty} \alpha_n \langle e_n, e_k\rangle_0 e_n \Big\|_U^2 = \sum_{k=1}^{\infty} \|\alpha_k e_k\|_U^2 \\ &= \sum_{k=1}^{\infty} \alpha_k^2 \|Q^{1/2}Q^{-1/2}e_k\|_U^2 \le \|Q^{1/2}\| \sum_{k=1}^{\infty} \alpha_k^2 < \infty. \end{aligned}$$

We recall the following elementary fact from real-valued probability theory.

Lemma 4.2. *Let $X \in L_1(\Omega, \mathcal{F}, P; \mathbf{R})$ be a random variable and $\mathcal{G}_2, \mathcal{G}_2 \subset \mathcal{F}$ be σ-algebras. If $\mathcal{G}_1$ is independent of $\sigma(\sigma(X) \cup \mathcal{G}_2)$, then*

$$\mathbf{E}\big(X \big| \sigma(\mathcal{G}_1 \cup \mathcal{G}_2)\big) = \mathbf{E}\big(X \big| \mathcal{G}_2\big).$$

Proposition 4.3 (Cylindrical Wiener process). *Let $\{e_k\}_{k\in\mathbf{N}}$ be an orthonormal basis of $U_0 = Q^{1/2}(U)$ and let $\{\beta_k\}_{k\in\mathbf{N}}$ be a family of independent real-valued Brownian motions. Let $(\tilde{U}, [\cdot,\cdot])$ with norm $[\![\cdot]\!]$ be a separable Hilbert space such that there is an embedding $J : U_0 \to \tilde{U}$ which is Hilbert-Schmidt. Then $\tilde{Q} : \tilde{U} \to \tilde{U}$ defined by $\tilde{Q} := JJ^*$ is bounded, $\tilde{Q} \ge 0$, $\mathrm{Tr}(\tilde{Q}) < \infty$, and the series*

$$\tilde{W}(t) = \sum_{k=1}^{\infty} \beta_k(t) J e_k, \quad t \in [0, T], \tag{4.1}$$

converges in $\mathcal{M}_T^2(\tilde{U})$ *and defines a* $\tilde{Q}$*-Wiener process on* $\tilde{U}$*. Moreover,*

$$\tilde{U_0} := \tilde{Q}^{1/2}(\tilde{U}) = J(U_0)$$

and, for all $u \in U_0$*,*

$$\|u\|_0 = [\![\tilde{Q}^{-1/2}Ju]\!] := [\![Ju]\!]_0.$$

That is, $J : U_0 \to \tilde{U_0}$ *is an isometric isomorphism.*

Proof. We show first that that $\{\tilde{W}(t)\}_{t\in[0,T]}$, defined in (4.1) is a $\tilde{Q}$-Wiener process on $\tilde{U}$. Let $\xi_j(t) = \beta_j(t)Je_j$, $j \in \mathbf{N}$, and define

$$\mathcal{G}_t := \sigma\Big(\bigcup_{j=1}^{\infty} \sigma(\{\beta_j(s)\}_{s\leq t})\Big), \quad t \in [0,T].$$

Then $\{\xi_j(t)\}_{t\in[0,T]}$ is a continuous $\tilde{U}$-valued martingale with respect to $\{\mathcal{G}_t\}_{t\geq 0}$ for all $j \in \mathbf{N}$. Indeed, take $0 \leq s \leq t \leq T$ and then

$$\mathbf{E}\Big(\beta_j(t)\Big|\mathcal{G}_s\Big) = \mathbf{E}\Big(\beta_j(t)\Big|\sigma(\{\beta_j(u)\}_{u\leq s}\Big) = \beta_j(s),$$

which follows from Lemma 4.2 with $X = \beta_j(t)$, $\mathcal{G}_2 = \sigma(\{\beta_j(u)\}_{u\leq s})$, and $\mathcal{G}_1 = \sigma\Big(\bigcup_{k\neq j}\{\beta_k(s)\}_{s\leq t}\Big)$. Therefore

$$\tilde{W}_n(t) := \sum_{j=1}^{n} \beta_j(t)Je_j, \quad t \in [0,T]$$

is also a continuous $\tilde{U}$-valued martingale with respect to $\{\mathcal{G}_t\}_{t\in[0,T]}$.

Moreover, since $J : U_0 \to \tilde{U}$ is a Hilbert-Schmidt operator,

$$\|\tilde{W}_m - \tilde{W}_n\|^2_{\mathcal{M}^2_T(\tilde{U})} = \sup_{t\in[0,T]} \mathbf{E}\Big(\llbracket \tilde{W}_m(t) - \tilde{W}_n(t) \rrbracket\Big)^2$$

$$= \mathbf{E}\Big(\llbracket \tilde{W}_m(T) - \tilde{W}_n(T)\rrbracket^2\Big) = \mathbf{E}\Big(\Big\llbracket \sum_{j=n+1}^{m} \beta_j(T) J e_j \Big\rrbracket^2\Big)$$

$$\leq \mathbf{E}\Big(\sum_{j=n+1}^{m} \beta_j(T)^2 \sum_{j=n+1}^{m} \llbracket J e_j \rrbracket^2\Big) = \sum_{j=n+1}^{m} \mathbf{E}\Big(\beta_j(T)^2\Big) \sum_{j=n+1}^{m} \llbracket J e_j \rrbracket^2$$

$$= T \sum_{j=n+1}^{m} \llbracket J e_j \rrbracket^2 \to 0, \quad n < m,\ n \to \infty.$$

Therefore, $\tilde{W}_n$ converges in $\mathcal{M}^2_T(\tilde{U})$ and its limit $\tilde{W} \in \mathcal{M}^2_T(\tilde{U})$ is continuous almost surely. The mean of W is clearly is zero. The increments are Gaussian since, for all $u \in \tilde{U}$,

$$[\tilde{W}(t) - \tilde{W}(s), u] = \sum_{j=1}^{\infty} (\beta_j(t) - \beta_j(s))[J e_j, u]$$

is Gaussian being an $L_2(\Omega, \mathcal{F}, P; \mathbf{R})$ limit of Gaussian random variables. To compute the covariance operator of the increments, take $u, v \in \tilde{U}$, $0 \leq s \leq t \leq T$ and write

$$\mathbf{E}\big([\tilde{W}(t) - \tilde{W}(s), u] \cdot [\tilde{W}(t) - \tilde{W}(s), v]\big) = \sum_{k=1}^{\infty} (t-s)[J e_k, u][J e_k, v]$$

$$= \sum_{k=1}^{\infty} (t-s)\langle e_k, J^* u\rangle_0 \langle e_k, J^* v\rangle_0 = (t-s)\langle J^* u, J^* u\rangle_0 = (t-s)[J J^* u, v],$$

where we used that $\mathbf{E}\big(\beta_j(t) - \beta_j(s))(\beta_k(t) - \beta_k(s)\big) = \delta_{jk}$. Thus, $\tilde{Q} = JJ^*$. One easily checks that the increments are independent. Finally, we have to show that

$$\tilde{Q}^{1/2}(\tilde{U}) = J(U_0)$$

and that

$$\|u\|_0 = [\![\tilde{Q}^{-1/2}Ju]\!] = [\![Ju]\!]_0, \quad \forall u \in U_0.$$

We recall the fact from functional analysis that if U_1, U_2 and H are separable Hilbert spaces, $T_1 \in L(U_1, H)$, $T_2 \in L(U_2, H)$, and

$$\|T_1^* x\|_1 = \|T_2^* x\|_2, \quad \forall x \in H,$$

then

$$T_1(U_1) = T_2(U_2)$$

and

$$\|T_1^{-1} x\|_1 = \|T_2^{-1} x\|_2, \quad \forall x \in T_1(U_1).$$

For all $u \in \tilde{U}$, we have that

$$[\![\tilde{Q}^{1/2}u]\!]^2 = [JJ^* u, u] = \|J^* u\|_0^2.$$

Thus, taking $U_1 = H = \tilde{U}$, $U_2 = U_0$, $T_1 = Q^{1/2}$, and $T_2 = J$, it follows that $\tilde{Q}^{1/2}(\tilde{U}) = J(U_0)$ and that $[\![\tilde{Q}^{-1/2}u]\!] = \|J^{-1}u\|_0$, for all $u \in J(U_0)$. Finally, if $v \in U_0$, then $[\![\tilde{Q}^{-1/2}Jv]\!] = \|v\|_0$ and hence $[\![Jv]\!]_0 = \|v\|_0$. □

Remark 4.4. The Wiener process constructed in Proposition 4.3 is independent of the choice of the orthonormal basis chosen for U_0. Indeed, the proof shows that with any such orthonormal basis, $\{\tilde{W}(t)\}_{t\in[0,T]}$ is a JJ^*-Wiener process on $\tilde{U}$. As such, by Proprosition 2.12, it can be obtained as (2.8), where the series converges in

$$L_2\big(\Omega, \mathcal{F}, P; C\big([0,T], U\big)\big).$$

Therefore, the paths of its limit are determined P-almost surely, that is, using two different orthonormal bases for U_0 we get two indistinguishable versions of $\{\tilde{W}(t)\}_{t\in[0,T]}$.

Remark 4.5. If $\mathrm{Tr}(Q) < \infty$, then one may choose $U = \tilde{U}$ and $J = I$. By Remark 4.4, the process $\{\tilde{W}(t)\}_{t\in[0,T]}$ is an indistinguishable version of the Q-Wiener process obtained by (2.8). The orthonormal basis $\{\lambda_k^{1/2} e_k : \lambda_k > 0\}$ used in (2.8) is just a particular choice of an orthonormal basis for U_0.

Now we are ready to define the stochastic integral with respect to a cylindrical Wiener process. Since $\mathrm{Tr}(\tilde{Q}) < \infty$, we can integrate processes $\{\Phi(t)\}_{t\in[0,T]}$ which are $L_2(\tilde{U}_0, H)$-predictable and

$$\mathbf{E}\Big(\int_0^T \|\Phi(s)\|^2_{L_2(\tilde{U}_0,H)}\,\mathrm{d}s\Big) < \infty.$$

But we are aiming at integrating processes with values in $L_2(U_0,H)$. We saw that U_0 is isometrically isomorphic to $\tilde{U}_0$ under J. Hence, if $\{e_k\}_{k\in\mathbf{N}}$ is an orthonormal basis for U_0, then $\{Je_k\}_{k\in\mathbf{N}}$ is an orthonormal basis for $\tilde{U}_0$. Therefore,

$$\Phi \in L_2(U_0,H) \quad \Leftrightarrow \quad \Phi J^{-1} \in L_2(\tilde{U}_0,H),$$

since

$$\begin{aligned}\|\Phi\|^2_{L_2(U_0,H)} &= \sum_{k=1}^{\infty} \langle \Phi e_k, \Phi e_k\rangle = \sum_{k=1}^{\infty} \langle \Phi J^{-1} J e_k, \Phi J^{-1} J e_k\rangle \\ &= \|\Phi J^{-1}\|^2_{L_2(\tilde{U}_0,H)}.\end{aligned}$$

Note that an $L_2(U_0,H)$-valued process is $\{\Phi(t)\}_{t\in[0,T]}$ is $L_2(U_0,H)$-predictable if and only if $\{\Phi(t)J^{-1}\}_{t\in[0,T]}$ is $L_2(\tilde{U}_0,H)$-predictable.

Let $Q \in L(U), Q \geq 0$, let $\{e_k\}_{k\in\mathbf{N}}$ be an orthonormal basis of $U_0 = Q^{1/2}(U)$ and let $\{\beta_k\}_{k\in\mathbf{N}}$ be a family of independent real valued Brownian motions. Define

$$W(t) = \sum_{k=1}^{\infty} e_k \beta_k(t), \quad t \in [0,T], \tag{4.2}$$

where the sum is understood only formally if $\mathrm{Tr}(Q) = \infty$.

Definition 4.6 (Integral with respect to a cylindrical Wiener process)**.** *Let* $\{W(t)\}_{t\in[0,T]}$ *be given by* (4.2)*. For processes* $\{\Phi(t)\}_{t\in[0,T]} \in \mathcal{N}_W^2$, *where*

$$\mathcal{N}_W^2 = \Big\{\Phi : [0,T] \times \Omega \to L_2(U_0, H),$$
$$\text{such that } \Phi \text{ is } L_2(U_0,H)\text{-predictable and } \|\Phi\|_T < \infty\Big\},$$

we define the stochastic integral by

$$\int_0^t \Phi(s)\,\mathrm{d}W(s) := \int_0^t \Phi(s)J^{-1}\,\mathrm{d}\tilde{W}(s), \quad t \in [0,T],$$

where the integral on the right hand side is the stochastic integral defined in Section 3 of $\{\Phi(t)J^{-1}\}_{t\in[0,T]}$ *with respect to the* $\tilde{U}$*-Wiener process* $\{\tilde{W}(t)\}_{t\in[0,T]}$ *defined in Proposition 4.3.*

Remark 4.7. If $\mathrm{Tr}(Q) = \infty$, then $\mathcal{E} \nsubseteq \mathcal{N}_W$, where $\mathcal{E}$ denotes the set of $L(U,H)$-valued elementary process from Definition 3.1. To see this, let $U = H, \Phi(t) \equiv I$, and $\{e_k\}$ be an orthonormal basis for U_0. Then

$$\|\Phi\|_T^2 = T\sum_{k=1}^{\infty}\langle e_k, e_k\rangle = T\sum_{k=1}^{\infty}\langle QQ^{-1/2}e_k, Q^{-1/2}e_k\rangle = T\,\mathrm{Tr}(Q) = \infty.$$

Remark 4.8. The cylindrical Wiener process $\{\tilde{W}(t)\}_{t\in[0,T]}$ constructed in Proposition 4.3 depends on J but $\int_0^t \Phi\,\mathrm{d}W$ does not. The proof is left to the reader as an exercise.

5 Stochastic evolution equations with additive noise

In the present section we introduce solution concepts to certain type of stochastic evolution problems and prove existence and uniqueness of their solutions. The mathematical framework is based on the theory of strongly continuous operator semigroups.

5.1 Linear equations

Let $\{W(t)\}_{t\in[0,T]}$ be an U-valued Q-Wiener process on the probability space $(\Omega,\mathcal{F},P)$, adapted to a normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. We consider equations written formally as

$$
\begin{aligned}
&\mathrm{d}X(t) = (AX(t)+f(t))\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad 0<t<T,\\
&X(0)=\xi,
\end{aligned}
\tag{5.1}
$$

where we make the following assumptions.

(A1) $A:\mathcal{D}(A)\subset H\to H$ is linear operator, generating a strongly continuous semigroup (C_0-semigroup) of bounded linear operators $\{S(t)\}_{t\geq 0}$, that is,

- $S(0)=I$;
- $S(t+s)=S(t)S(s)$ for all $s,t\geq 0$;
- $\{S(t)\}_{t\geq 0}$ is strongly continuous on $[0,\infty)$, that is, $t\mapsto S(t)x$ is continuous on $[0,\infty)$ for all $x\in H$;
- $\lim_{h\to 0^+}\frac{S(t+h)x-S(t)x}{h}=Ax$ for all $x\in\mathcal{D}(A)$;

(A2) $B \in L(U,H)$;

(A3) $\{f(t)\}_{t\in[0,T]}$ a predictable H-valued process with Bochner integrable trajectories, that is, $t \mapsto f(\omega,t)$ is Bochner integrable on $[0,T]$ for P-almost all $\omega \in \Omega$;

(A4) ξ is an $\mathcal{F}_0$-measurable H-valued random variable.

Under assumption (A1) the deterministic evolution problem (abstract Cauchy problem)

$$\begin{aligned} &u'(t) = Au(t) + f(t), \quad t > 0, \\ &u(0) = x, \end{aligned}$$

is well-posed (under some weak assumptions on f) and its unique (mild) solution is given by the variation of constants formula

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,\mathrm{d}s.$$

For an exhaustive introduction to the theory of operator semigroups, see, for example, [1] and [5].

Remark 5.1. Since H is, in particular, a reflexive Banach space it follows that $\{S(t)^*\}_{t\geq 0}$ is also a C_0-semigroup on H with generator given by A^*, the adjoint of A. In non-reflexive Banach spaces this is not true in general.

Next we discuss what we mean by the solution of the formal equation (5.1). In this section we always assume (A1)–(A4).

Definition 5.2 (Strong solution)**.** *An H-valued process $\{X(t)\}_{t\in[0,T]}$ is a strong solution of* (5.1) *if $\{X(t)\}_{t\in[0,T]}$ is H-predictable, $X(t,\omega) \in$*

$\mathcal{D}(A)$ P_T*-almost surely,* $\int_0^T \|AX(t)\|\,\mathrm{d}t < \infty$ *P-almost surely, and, for all* $t \in [0,T]$,

$$X(t) = \xi + \int_0^t \big(AX(s) + f(s)\big)\,\mathrm{d}s + \int_0^t B\,\mathrm{d}W(s), \quad P\text{-a.s.}$$

Recall that the integral $\int_0^t B\,\mathrm{d}W(s)$ is defined if and only if $\|B\|_{L_2^0}^2 = \mathrm{Tr}(BQB^*) < \infty$.

For $\eta \in H$, we define

$$l_\eta : H \to \mathbf{R}, \quad l_\eta(h) := \langle h, \eta \rangle, \; h \in H. \tag{5.2}$$

Definition 5.3 (Weak solution). *An H-valued process* $\{X(t)\}_{t\in[0,T]}$ *is a weak solution of* (5.1) *if* $\{X(t)\}_{t\in[0,T]}$ *is H-predictable,* $\{X(t)\}_{t\in[0,T]}$ *has Bochner integrable trajectories P-almost surely and*

$$\begin{aligned}\langle X(t), \eta \rangle = \langle \xi, \eta \rangle &+ \int_0^t \big(\langle X(s), A^*\eta \rangle + \langle f(s), \eta \rangle\big)\,\mathrm{d}s \\ &+ \int_0^t l_\eta B\,\mathrm{d}W(s), \quad P\text{-a.s.}, \; \forall \eta \in \mathcal{D}(A), \; t \in [0,T].\end{aligned}$$

Note that the stochastic integral may be written formally as

$$\int_0^t l_\eta B\,\mathrm{d}W(s) = \int_0^t \langle B\,\mathrm{d}W(s), \eta \rangle.$$

We will show that unique weak solution of (5.1) is given by the variation of constants formula

$$X(t) = S(t)\xi + \int_0^t S(t-s)f(s)\,\mathrm{d}s + \int_0^t S(t-s)B\,\mathrm{d}W(s).$$

We will need the following lemma about interchanging the stochastic integral with closed operators.

Lemma 5.4. *Let E be a separable Hilbert space. Let $\Phi \in \mathcal{N}_W^2$, $A : \mathcal{D}(A) \subset H \to E$ be a closed, linear operator with $\mathcal{D}(A)$ being a Borel subset of H. If $\Phi(t)u \in \mathcal{D}(A)$ P-almost surely for all $t \in [0,T]$ and $u \in U$ and $A\Phi \in \mathcal{N}_W^2$, then*

$$P\Big(\int_0^T \Phi(s)\,\mathrm{d}W(s) \in \mathcal{D}(A)\Big) = 1$$

and

$$A\Big(\int_0^T \Phi(s)\,\mathrm{d}W(s)\Big) = \int_0^T A\Phi(s)\,\mathrm{d}W(s), \quad P\text{-a.s.} \tag{5.3}$$

Proof. The lemma is a special case of [4, Proposition 4.15]. □

Note that if $A \in L(H,E)$, then (5.3) holds for all $\Phi \in \mathcal{N}_W^2$. We define the stochastic convolution

$$W_A(t) := \int_0^t S(t-s)B\,\mathrm{d}W(s)$$

and the operator

$$Q_t = \int_0^t S(s)BQB^*S(s)^*\,\mathrm{d}s,$$

where the integral is a strong Bochner integral. The following theorem provides the basic properties of the stochastic convolution.

Theorem 5.5. *If for some $T > 0$,*

$$\int_0^T \|S(t)B\|_{L_2^0}^2\,\mathrm{d}s = \int_0^T \mathrm{Tr}(S(t)BQB^*S(t)^*)\,\mathrm{d}t = \mathrm{Tr}(Q_T) < \infty,$$

then

1. *$W_A \in C\big([0,T], L_2(\Omega,\mathcal{F},P;H)\big)$ and W_A has an H-predictable version;*

2. $\{W_A(t)\}_{t\in[0,T]}$ *is a Gaussian process and*

$$\mathrm{Cov}(W_A(t)) = \int_0^t S(s)BQB^*S(s)^*\,\mathrm{d}s = Q_t.$$

Proof. Let $0 \le s \le t \le T$ and define

$$\Phi(r) = S(t-r)B, \quad M_t(s) = \int_0^s \Phi\,\mathrm{d}W = \int_0^s S(t-r)B\,\mathrm{d}W(r).$$

Then

$$\begin{aligned}\mathbf{E}\int_0^t \|\Phi\|^2_{L_2^0}\,\mathrm{d}r &= \int_0^t \|S(t-r)B\|^2_{L_2^0}\,\mathrm{d}r = \int_0^t \|S(r)B\|^2_{L_2^0}\,\mathrm{d}r \\ &\le \int_0^T \|S(r)B\|^2_{L_2^0}\,\mathrm{d}r < \infty.\end{aligned}$$

Thus, $M_t(s)$ is well defined, in particular, for $s=t$ it follows that $M_t(t) = W_A(t)$ is well defined. To show mean square continuity, let $0 \le s \le t \le T$. Then

$$\begin{aligned}W_A(t) - W_A(s) &= \int_0^t S(t-r)B\,\mathrm{d}W(r) - \int_0^s S(s-r)B\,\mathrm{d}W(r) \\ &= \int_0^s \big(S(t-r) - S(s-r)\big)B\,\mathrm{d}W(r) \\ &\quad + \int_0^t 1_{(s,t]}S(t-r)B\,\mathrm{d}W(r) = X + Y.\end{aligned} \tag{5.4}$$

The random variables X and Y are independent with zero mean and therefore, using also Itô's isometry,

$$\begin{aligned}\mathbf{E}(\|W_A(t) - W_A(s)\|^2) &= \mathbf{E}\Big(\Big\|\int_0^s \big(S(t-s) - I\big)S(s-r)B\,\mathrm{d}W(r)\Big\|^2\Big) \\ &\quad + \mathbf{E}\Big(\Big\|\int_0^t 1_{(s,t]}S(t-r)B\,\mathrm{d}W(r)\Big\|^2\Big) \\ &= \int_0^s \|(S(t-s) - I)S(r)BQ^{1/2}\|^2_{L_2(U,H)}\,\mathrm{d}r \\ &\quad + \int_0^{t-s} \|S(r)BQ^{1/2}\|^2_{L_2(U,H)}\,\mathrm{d}r \to 0 \quad \text{as } s \to t.\end{aligned}$$

The second integral converges 0 by the Dominated Convergence Theorem. For the first one, we have

$$\begin{aligned}1_{(0,s]}(r)\|(S(t-s)-I)S(r)BQ^{1/2}\|^2_{L_2(U,H)}\\ \le 2\max_{0\le s\le T}\|S(s)\|^2_{L(H)}\|S(r)BQ^{1/2}\|^2_{L_2(U,H)},\end{aligned}$$

and therefore we may use again dominated convergence together with the fact that $S(t-s)-I\to 0$ strongly as $t-s\to 0$.

For the existence of a predictable version of $\{W_A(t)\}_{t\in[0,T]}$ note that if $\{X(t)\}_{t\in[0,T]}$ is mean square continuous, then it is uniformly stochastically continuous[2] on $[0,T]$. This follows from the observation that the mean square continuity of $\{X(t)\}_{t\in[0,T]}$ means that $X(\cdot)$ is continuous as a function $[0,T]\to L_2(\Omega,\mathcal{F},P;H)$. Since $[0,T]$ is compact $\{X(t)\}_{t\in[0,T]}$ is uniformly mean square continuous on $[0,T]$. We have that

$$P(\|X(t)-X(s)\|^2\ge\varepsilon^2)\le\frac{1}{\varepsilon^2}\mathbf{E}(\|X(t)-X(s)\|^2)$$

and hence $\{X(t)\}_{t\in[0,T]}$ is uniformly stochastically continuous on $[0,T]$. By [4, Proposition 3.6], $\{X(t)\}_{t\in[0,T]}$ has a predictable version since it is clearly adapted and stochastically continuous.

For t fixed, the random variable $W_A(t)$ is Gaussian. This follows from the construction of the integral and the fact that for elementary deterministic processes the stochastic integral is a Gaussian random variable. An easy calculation shows, similar to the one in (5.4), that for all $u_1,u_2,\dots,u_n\in U$, $(\langle W_A(t_1),u_1\rangle,\dots,\langle W_A(t_n),u_n\rangle)$ is an $\mathbf{R}^n$-valued Gaussian random variable using also Lemma 5.4 for $A=l_{u_i},\ i=1,\dots,n$. Fi-

[2] A process $\{X(t)\}_{t\in[0,T]}$ is uniformly stochastically continuous on $[0,T]$ if $\forall\varepsilon>0,\ \forall\delta>0,\ \exists\gamma>0,$ such that $P(\|X(t)-X(s)\|\ge\varepsilon)\le\delta,\ |t-s|<\gamma,\ t,s\in[0,T]$.

nally, the covariance operator Q_t of $W_A(t)$ can be computed in a straightforward fashion using Lemma 5.4, Corollary 3.6 and Parseval's formula. □

Before proving the existence and uniqueness of weak solutions of (5.1) we need a few preparatory results which we state with only a reference to the proofs. Consider the following assumptions.

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration. Let $\Phi \in \mathcal{N}_W^2[0,T]$, ϕ be an H-valued predictable process, Bochner integrable on $[0,T]$ P-almost surely, and $X(0)$ be an $\mathcal{F}_0$-measurable H-valued random variable.

2. Let $F : [0,T] \times H \to \mathbf{R}$ and assume that the Fréchet derivatives $F_t(t,x)$, $F_x(t,x)$, and $F_{xx}(t,x)$ are uniformly continuous as functions of (t,x) on bounded subsets of $[0,T] \times H$. Note that, for fixed t, $F_x(t,x) \in L(H,\mathbf{R})$ and we consider $F_{xx}(t,x)$ as an element of $L(H)$.

Theorem 5.6 (Itô's formula)**.** *Under assumptions (1) and (2) above, let*

$$X(t) = X(0) + \int_0^t \phi(s)\,\mathrm{d}s + \int_0^t \Phi(s)\,\mathrm{d}W(s), \quad t \in [0,T].$$

Then, P-almost surely and for all $t \in [0,T]$,

$$\begin{aligned} F(t,X(t)) = F(0,X(0)) &+ \int_0^t F_x(s,X(s))\Phi(s)\,\mathrm{d}W(s) \\ &+ \int_0^t \Big(F_t(s,X(s)) + F_x(s,X(s))(\phi(s)) \\ &\quad + \tfrac{1}{2}\operatorname{Tr}\big((F_{xx}(s,X(s)))(\Phi(s)Q^{1/2})(\Phi(s)Q^{1/2})^*\big)\Big)\,\mathrm{d}s. \end{aligned}$$

Proof. See [4, Theorem 4.17]. □

The next result is the stochastic version of Fubini's Theorem. Consider the following.

(3) Let $(E,\mathcal{E})$ be a measurable space and

$$\Phi : (\Omega_T \times E, \mathcal{P}_T \times \mathcal{E}) \to (L_2^0, \mathcal{B}(L_2^0))$$

be a measurable mapping.

(4) Let μ be a finite positive measure on $(E,\mathcal{E})$.

(5) Assume that $\int_E \|\Phi(\cdot,\cdot,x)\|_T \,\mathrm{d}\mu(x) < \infty$.

Note, that, in particular, for fixed $x \in E$, the process $\Phi(\cdot,\cdot,x)$ is L_2^0-predictable and $\Phi(\cdot,\cdot,x) \in \mathcal{N}_W^2[0,T]$.

Theorem 5.7 (Stochastic Fubini's Theorem)**.** *Assuming (3)–(5) above, we have P-almost surely,*

$$\int_E \int_0^T \Phi(t,x)\,\mathrm{d}W(t)\,\mathrm{d}\mu(x) = \int_0^T \int_E \Phi(t,x)\,\mathrm{d}\mu(x)\,\mathrm{d}W(t). \tag{5.5}$$

Proof. See [4, Theorem 4.18]. □

Note that the inner integral on the right hand side of (5.5) is an L_2^0-valued Bochner integral. Now we can the prove existence of weak solutions of (5.1). Let

$$X(t) := S(t)\xi + \int_0^t S(t-s)f(s)\,\mathrm{d}s + W_A(t) = Y(t) + W_A(t). \tag{5.6}$$

Theorem 5.8 (Existence of weak solutions)**.** *Assume (A1)–(A4) and*

$$\int_0^T \|S(r)B\|^2_{L_2^0}\,\mathrm{d}r < \infty.$$

Then $\{X(t)\}_{t\in[0,T]}$ *defined in* (5.6) *has a version which is a weak solution of* (5.1).

Proof. The process $\{X(t)\}_{t\in[0,T]}$ has Bochner integrable trajectories and an H-predictable version by Theorem 5.5. Since $\{Y(t)\}_{t\in[0,T]}$ is the (unique) weak solution of

$$\begin{aligned} Y'(t) &= AY(t) + f(t), \quad t > 0, \\ Y(0) &= \xi, \end{aligned}$$

it follows that $\{X(t)\}$ is a weak solution of (5.3) if and only if $W_A(t) = X(t) - Y(t)$ is a weak solution of

$$\begin{aligned} \mathrm{d}X(t) &= AX(t)\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad 0 < t < T, \\ X(0) &= 0. \end{aligned} \tag{5.7}$$

Therefore, without loss of generality, we may set $\xi = 0$, $f = 0$ and show that $W_A(t)$ is a weak solution of (5.7). If $t \in [0,T]$ and $\eta \in \mathcal{D}(A^*)$, then

$$\int_0^t \langle A^*\eta, W_A(s)\rangle\,\mathrm{d}s = \int_0^t \Big\langle A^*, \int_0^t 1_{[0,s]}(r)S(s-r)B\,\mathrm{d}W(r)\Big\rangle\,\mathrm{d}s.$$

Following (5.2), we set $l_{A^*\eta}(u) := \langle A^*\eta, u\rangle$. Then, by Lemma 5.4 and Theorem 5.7,

$$\begin{aligned} \int_0^t \langle A^*\eta, W_A(t)\rangle\,\mathrm{d}s &= \int_0^t l_{A^*\eta}\Big(\int_0^t 1_{[0,s]}(r)S(s-r)B\,\mathrm{d}W(r)\Big)\,\mathrm{d}s \\ &= \int_0^t \int_0^t 1_{[0,s]}(r)l_{A^*\eta}S(s-r)B\,\mathrm{d}W(r)\,\mathrm{d}s \\ &= \int_0^t \int_0^t 1_{[0,s]}(r)l_{A^*\eta}S(s-r)B\,\mathrm{d}s\,\mathrm{d}W(r) \\ &= \int_0^t \int_r^t l_{A^*\eta}S(s-r)B\,\mathrm{d}s\,\mathrm{d}W(r). \end{aligned}$$

For all $u \in U$,

$$l_{A^*\eta} S(s-r)Bu = \langle A^*\eta, S(s-r)Bu\rangle = \langle S(s-r)^* A^*\eta, Bu\rangle,$$

and hence, using that $\eta \in \mathcal{D}(A^*)$,

$$\begin{aligned}
\int_r^t l_{A^*\eta} S(s-r)Bu \, \mathrm{d}s &= \int_r^t \langle S(s-r)^* A^*\eta, Bu\rangle \, \mathrm{d}s \\
&= \int_r^t \langle A^* S(s-r)^*\eta, Bu\rangle \, \mathrm{d}s \\
&= \int_r^t \frac{\mathrm{d}}{\mathrm{d}s} \langle S(s-r)^*\eta, Bu\rangle \, \mathrm{d}s \\
&= \langle \eta, S(s-r)Bu\rangle - \langle \eta, Bu\rangle .
\end{aligned}$$

Finally, by Lemma 5.4,

$$\begin{aligned}
\int_0^t \int_0^t 1_{[0,s]}(r) l_{A^*\eta} S(s-r)B \, \mathrm{d}s \, \mathrm{d}W(r) &= \int_0^t l_\eta S(t-r)B \, \mathrm{d}W(s) \\
- \int_0^t l_\eta B \, \mathrm{d}W(s) &= \langle \eta, W_A(t)\rangle - \int_0^t l_\eta B \, \mathrm{d}W(s), \ P\text{-a.s.}
\end{aligned}$$

□

To prove uniqueness of weak solutions of (5.1) we need the following two results.

Lemma 5.9. *Let $(C, \mathcal{D}(C))$ be the generator of a C_0-semigroup on the separable Hilbert space H. Then, the vector space $\mathcal{D}(C)$ endowed with inner product $\langle x,y\rangle_C := \langle x,y\rangle_H + \langle Cx,Cy\rangle_H$ and norm $\|x\|_C := \langle x,x\rangle_C^{1/2}$ is a separable Hilbert space.*

The proof is left to the reader as a (non-trivial) exercise.

Proposition 5.10. *Let* $\{X(t)\}_{t\geq 0}$ *be a weak solution of* (5.1) *with* $f=0$ *and* $\xi=0$*. Then, for all* $\rho \in C^1([0,T],\mathcal{D}(A^*))$ *and* $t\in[0,T]$*,*

$$\langle X(t),\rho(t)\rangle = \int_0^t \langle X(s),\rho'(s)+A^*\rho(s)\rangle\,\mathrm{d}s + \int_0^t l_{\rho(s)}B\,\mathrm{d}W(s).$$

Proof. First, let $\rho(s):=\rho_0\phi(s)$, $\rho_0\in\mathcal{D}(A^*)$, $\phi\in C^1([0,T],\mathbf{R})$ and define

$$Y_{\rho_0}(t) := \int_0^t \langle X(s),A^*\rho_0\rangle\,\mathrm{d}s + \int_0^t l_{\rho_0}B\,\mathrm{d}W(s).$$

Note that if $\{X(t)\}_{t\in[0,T]}$ is a weak solution with $f=0$ and $\xi=0$, then

$$\langle X(t),\rho_0\rangle = Y_{\rho_0}(t), \quad t\in[0,T]. \tag{5.8}$$

If $F(t,x):=\phi(t)x$, $x\in\mathbf{R}$, $t\in[0,T]$, then

$$F_t(t,x)=x\phi'(t), \quad F_x(t,x)=\phi(t), \quad F_{xx}(t,x)=0,$$

and hence, by Theorem 5.6 and (5.8),

$$\begin{aligned}\langle X(t),\rho(t)\rangle &= \phi(t)\langle X(t),\rho_0\rangle = \phi(t)Y_{\rho_0}(t) = F(t,Y_{\rho_0}(t))\\ &= \int_0^t \phi(s)l_{\rho_0}B\,\mathrm{d}W(s) + \int_0^t \big(Y_{\rho_0}(s)\phi'(s)+\phi(s)\langle X(s),A^*\rho_0\rangle\big)\,\mathrm{d}s\\ &= \int_0^t l_{\rho(s)}B\,\mathrm{d}W(s) + \int_0^t \langle X(s),\rho'(s)+A^*\rho(s)\rangle\,\mathrm{d}s.\end{aligned}$$

Next consider a general $\rho\in C^1([0,T],\mathcal{D}(A^*))$. By Remark 5.1 the operator A^* is the generator of the C_0-semigroup $\{S(t)^*\}_{t\geq 0}$ and hence, by Lemma 5.9, $\mathcal{D}(A^*)$ becomes a separable Hilbert space with inner product $\langle x,y\rangle_{A^*} := \langle x,y\rangle_H + \langle A^*x,A^*y\rangle_H$ and norm $\|x\|_{A^*} := \langle x,x\rangle_{A^*}^{1/2}$. Let $\{e_k\}_{k\in\mathbf{N}}$ be an orthonormal basis for $(\mathcal{D}(A^*),\|\cdot\|_{A^*})$ and consider the orthogonal expansions

$$\rho(t)=\sum_{k=1}^{\infty}\langle\rho(t),e_k\rangle_{A^*}e_k \quad\text{and}\quad \rho'(t)=\sum_{k=1}^{\infty}\langle\rho'(t),e_k\rangle_{A^*}e_k.$$

For $N \in \mathbf{N}$, define

$$\rho_N(t) := \sum_{k=1}^{N} \langle \rho(t), e_k \rangle_{A^*} e_k, \quad \rho'_N(t) = \sum_{k=1}^{N} \langle \rho'(t), e_k \rangle_{A^*} e_k.$$

Then, by the first part of the proof and linearity,

$$\begin{aligned} \langle X(t), \rho_N(t) \rangle_H = \int_0^t \big(\langle X(s), \rho'_N(s) \rangle_H + \langle X(s), A^* \rho_N(s) \rangle_H \big) \,\mathrm{d}t \\ + \int_0^t l_{\rho_N(s)} B \,\mathrm{d}W(s). \end{aligned} \tag{5.9}$$

For the second integral on the right hand side of (5.9) we have, using Itô's isometry, that

$$\mathbf{E} \Big\| \int_0^t l_{\rho_N(s)} B \,\mathrm{d}W(s) - \int_0^t l_{\rho(s)} B \,\mathrm{d}W(s) \Big\|^2 \to 0,$$

since, by the Dominated Convergence Theorem,

$$\begin{aligned} &\int_0^t \| l_{\rho_N(s)} - l_{\rho(s)}) B Q^{1/2} \|^2_{L_2(U,\mathbf{R})} \,\mathrm{d}s \\ &\qquad = \int_0^t \| Q^{1/2} B^* (\rho_N(s) - \rho(s)) \|^2_U \,\mathrm{d}s \to 0. \end{aligned}$$

Finally, we may select a subsequence $\{\rho_{N_k}\}$ such that

$$\int_0^t l_{\rho_{N_k}(s)} B \,\mathrm{d}W(s) \to \int_0^t l_{\rho(s)} B \,\mathrm{d}W(s) \quad P\text{-almost surely, as } k \to \infty.$$

For the sake of simplicity we denote the sequence $\{\rho_{N_k}\}$ by $\{\rho_N\}$ again. To deal with the first integral on the right hand side of (5.9), we note that $\rho_N(t)$ and $\rho'_N(t)$ converge in the $\|\cdot\|_{A^*}$-norm to $\rho(t)$ and $\rho'(t)$, respectively. Hence, it follows that

$$\langle X(t), \rho_N(t) \rangle_H \to \langle X(t), \rho(t) \rangle_H,$$

$$\langle X(s), \rho_N'(s)\rangle_H \to \langle X(s), \rho'(s)\rangle_H,$$

and

$$\langle X(s), A^*\rho_N(s)\rangle_H \to \langle X(s), A^*\rho(s)\rangle_H$$

as $N \to \infty$. We also have

$$\begin{aligned} |\langle X(s), \rho_N'(s)\rangle|^2 &\le \|X(s)\|_H^2 \|\rho_N'(s)\|_H^2 \le \|X(s)\|_H^2 \|\rho_N'(s)\|_{A^*}^2 \\ &\le \|X(s)\|_H^2 \|\rho'(s)\|_{A^*}^2 \le K\|X(s)\|_H^2, \end{aligned}$$

and thus,

$$|\langle X(s), \rho_N'(s)\rangle| \le K\|X(s)\|_H. \tag{5.10}$$

Similarly, for the other term,

$$|\langle X(s), A^*\rho_N(s)\rangle| \le \cdots \le \|X(s)\|_H \|\rho(s)\|_{A^*} \le K\|X(s)\|_H. \tag{5.11}$$

Since $\{X(t)\}_{t\in[0,T]}$ is a weak solution of (5.1) it has Bochner integrable trajectories P-almost surely and hence, by (5.10), (5.10), and the Dominated Convergence Theorem, we may pass to the limit in (5.9) inside the first integral on the right hand side P-almost surely and the proof is complete. □

Theorem 5.11 (Uniqueness). *If $\{X(t)\}_{t\in[0,T]}$ is a weak solution of* (5.1)*, then $X(t)$ is given by* (5.6) *P-almost surely, that is, $\{X(t)\}_{t\in[0,T]}$ is a version of* (5.6).

Proof. As in the proof of existence of weak solutions of (5.1) it suffices to consider the case when $f = 0$ and $\xi = 0$. Let

$$\rho(s) := S(t-s)^*\rho_0, \quad s \in [0,T], \quad \rho_0 \in \mathcal{D}((A^*)^2).$$

Then $\rho'(s) = -A^*S(t-s)^*\rho_0 = -A^*\rho(s)$ and by Lemma 5.10,

$$\langle X(t), \rho_0\rangle = \langle X(t), \rho(t)\rangle = \int_0^t l_{\rho(s)} B \, \mathrm{d}W(s).$$

Furthermore,

$$(l_{\rho(s)}B)(u) = \langle S(t-s)^*\rho_0, Bu\rangle = (l_{\rho_0}S(t-s)B)(u)$$

and hence, by Lemma 5.4,

$$\begin{aligned}\langle X(t), \rho_0\rangle &= \int_0^t l_{\rho(s)} B \, \mathrm{d}W(s) = \int_0^t l_{\rho_0} S(t-s) B \, \mathrm{d}W(s)\\ &= l_{\rho_0}\Big(\int_0^t S(t-s)B \, \mathrm{d}W(s)\Big) = \langle W_A(t), \rho_0\rangle.\end{aligned}$$

Finally, using the fact from semigroup theory that $\mathcal{D}((A^*)^2)$ is dense in H, we conclude that $X(t) = W_A(t)$ P-almost surely. □

5.2 Semilinear equations with globally Lipschitz nonlinearity

As before, let $\{W(t)\}_{t\in[0,T]}$ be an U-valued Q-Wiener process on the probability space $(\Omega, \mathcal{F}, P)$, adapted to a normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. Here we consider equations written formally as

$$\begin{aligned}&\mathrm{d}X(t) = (AX(t) + f(X(t)))\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad 0 < t < T,\\ &X(0) = \xi.\end{aligned} \tag{5.12}$$

The main difference when dealing with this kind of equations compared to the one before is that, in general, there is no explicit representation of the solution of (5.12). We need another solution concept.

Definition 5.12 (Mild solution)**.** *An H-valued process $\{X(t)\}_{t\in[0,T]}$ is a mild solution of* (5.12) *if $\{X(t)\}_{t\in[0,T]}$ is adapted,*

$$X \in C\big([0,T];L_2(\Omega,\mathcal{F},P;H)\big)$$

and, for all $t \in [0,T]$,

$$X(t) = S(t)\xi + \int_0^t S(t-s)f(X(s))\,\mathrm{d}s + \int_0^t S(t-s)B\,\mathrm{d}W(s) \quad P\text{-a.s.}$$

Lemma 5.13. *Let $0 \le a < b$. The space*

$$Z_{[a,b]} := \Big\{X \in C\big([a,b];L_2(\Omega,\mathcal{F},P;H)\big) : X \text{ is adapted}\Big\}$$

with norm $\|Y\|_{Z_{[a,b]}} = \sup_{t\in[a,b]}(\mathbf{E}\|Y(t)\|_H^2)^{1/2}$ is a Banach space.

Proof. Exercise. *Hint:* show that $Z_{[a,b]}$ is a closed subspace of

$$C\big([a,b];L_2(\Omega,\mathcal{F},P;H)\big).$$

□

Theorem 5.14. *Let $A : \mathcal{D}(A) \subset H \to H$ be a linear operator, generating a C_0-semigroup $\{S(t)\}_{t\ge0}$ on H. Assume that $B \in L(U,H)$, $\xi \in L_2(\Omega,\mathcal{F}_0,P;H)$,*

$$\int_0^T \|S(s)BQ^{1/2}\|^2_{L_2(U,H)}\,\mathrm{d}s < \infty$$

and that $f : H \to H$ satisfies the global Lipschitz condition

$$\|f(x)-f(y)\|_H \le K\|x-y\|_H, \quad \forall x,y \in H,$$

for some $K > 0$. Then, there is a unique mild solution of (5.12).

Proof. Define

$$F(Y)(t) := S(t)\xi + \int_0^t S(t-s)f(Y(s))\mathrm{d}s + \int_0^t S(t-s)B\,\mathrm{d}W(s),$$
$$G(Y)(t) := \int_0^t S(t-s)f(Y(s))\mathrm{d}s.$$

We will show that the equation $X = F(X)$ in $Z_{[0,T]}$, where $Z_{[0,T]}$ is defined in Lemma 5.13, has a unique solution by Banach's fixed point theorem.

(1) Let $\tau > 0$. It is not difficult to see that $F : Z_{[0,\tau]} \to Z_{[0,\tau]}$, that is, $S(\cdot)\xi$, $G(Y)$, and W_A are mean square continuous and adapted on $[0,\tau]$. This is left as an exercise.

(2) To show that F is a contraction on $Z_{[0,\tau]}$ for some τ, we consider

$$\begin{aligned}
&\mathbf{E}\big(\|F(Y_1)(t) - F(Y_2)(t)\|^2\big)\\
&\quad = \mathbf{E}\Big(\Big\|\int_0^t S(t-s)(f(Y_1(s)) - f(Y_2(s)))\,\mathrm{d}s\Big\|^2\Big)\\
&\quad \le \mathbf{E}\Big(\Big(\int_0^t \|S(t-s)\|_{L(H)}\|f(Y_1(s)) - f(Y_2(s))\|_H\,\mathrm{d}s\Big)^2\Big).
\end{aligned}$$

With

$$M_T := \sup_{t\in[0,T]} \|S(t)\|_{L(H)}$$

and the Lipschitz condition, we obtain

$$\begin{aligned}
&\mathbf{E}\big(\|F(Y_1)(t)-F(Y_2)(t)\|^2\big)\\
&\quad\le M_T^2K^2\mathbf{E}\Big(\big(\int_0^\tau \|Y_1(s)-Y_2(s)\|\,\mathrm{d}s\big)^2\Big)\\
&\quad\le M_T^2K^2\mathbf{E}\Big(\tau\int_0^\tau \|Y_1(s)-Y_2(s)\|^2\,\mathrm{d}s\Big)\\
&\quad\le M_T^2K^2\tau\mathbf{E}\Big(\int_0^\tau \|Y_1(s)-Y_2(s)\|^2\,\mathrm{d}s\Big)\\
&\quad= M_T^2K^2\tau\int_0^\tau \underbrace{\mathbf{E}\big(\|Y_1(s)-Y_2(s)\|^2\big)}_{\le\sup_{[0,\tau]}\mathbf{E}(\|Y_1-Y_2\|^2)}\,\mathrm{d}s\\
&\quad\le M_T^2K^2\tau^2\|Y_1-Y_2\|^2_{Z_{[0,\tau]}}.
\end{aligned}$$

Thus,

$$\|F(Y_1)-F(Y_2)\|_{Z_{[0,\tau]}}\le M_TK\tau\|Y_1-Y_2\|_{Z_{[0,\tau]}}.$$

Choose τ so that $M_TK\tau<1$. Note that τ can be chosen independently of ξ. Then $F:Z_{[0,\tau]}\to Z_{[0,\tau]}$ is a contraction and therefore, by Banach's fixed point theorem, F has a unique fixed point $X_1\in Z_{[0,\tau]}$, which is the unique mild solution of (5.12) on $[0,\tau]$.

(3) Now consider the equation

$$Y(t)=S(t-\tau)X_1(\tau)+\int_\tau^t S(t-s)f(Y(s))\,\mathrm{d}s+\int_\tau^t S(t-s)B\,\mathrm{d}W(s),$$

where $t\in[\tau,2\tau]$. As above, we get an unique fixed point $Y\in Z_{[\tau,2\tau]}$. It is important here that the length τ of the interval can be chosen independently of the initial value $X_1(\tau)$. Define

$$X(t):=\begin{cases}X_1(t), & t\in[0,\tau],\\ Y(t), & t\in[\tau,2\tau].\end{cases}$$

Then $X \in Z_{[0,2\tau]}$ and for $t \in [\tau, 2\tau]$ we have that

$$\begin{aligned}
X(t) &= Y(t) \\
&= S(t-\tau)\Big(S(\tau)\xi + \int_0^\tau S(\tau - s)f(X(s))\,\mathrm{d}s + \int_0^\tau S(\tau - s)B\,\mathrm{d}W(s)\Big) \\
&\qquad + \int_\tau^t S(t-s)f(X(s))\,\mathrm{d}s + \int_\tau^t S(t-s)B\,\mathrm{d}W(s) \\
&= S(t)\xi + \int_0^t S(t-s)f(X(s))\,\mathrm{d}s + \int_0^t S(t-s)B\,\mathrm{d}W(s).
\end{aligned}$$

Thus, X is the unique mild solution of (5.12) on $[0, 2\tau]$. By repeating the above procedure a finite number of times, we obtain the unique mild solution of (5.12) on $[0, T]$. □

Remark 5.15. Since the mild solution of (5.12) is mean square continuous and adapted, it has a predictable version, c.f., the proof of Theorem 5.5. Also, as in the case of linear equations with additive noise, the solution is unique up to modification.

6 Examples

In this section we apply the abstract framework to the stochastic heat and wave equations driven by additive noise.

6.1 The heat equation

Let $\mathcal{D} \subset \mathbf{R}^d$, $d = 1,2,3$, be a spatial domain with smooth boundary $\partial\mathcal{D}$ and consider the stochastic heat equation

$$
\begin{aligned}
&\mathrm{d}X(\xi,t) = \Delta X(\xi,t)\,\mathrm{d}t + \mathrm{d}W(\xi,t), && \xi \in \mathcal{D},\ t > 0,\\
&X(\xi,t) = 0, && \xi \in \partial\mathcal{D},\ t > 0, \qquad (6.1)\\
&X(\xi,0) = X_0(\xi),
\end{aligned}
$$

where $\Delta = \sum_{k=1}^{d} \partial/\partial\xi_k^2$ denotes the Laplace operator. In order to put the equation into the semigroup framework of the previous section we define $H = U = L_2(\mathcal{D})$ and recall the Sobolev spaces

$$
\begin{aligned}
H^k &= H^k(\mathcal{D}) = \Big\{ v \in L_2(\mathcal{D}) : D^\alpha v \in L_2(\mathcal{D}),\ |\alpha| \le k \Big\},\\
H_0^1 &= H_0^1(\mathcal{D}) = \Big\{ v \in H^1(\mathcal{D}) : v|_{\partial\mathcal{D}} = 0 \Big\}.
\end{aligned}
$$

We consider $A = -\Delta$ as an unbounded linear operator on H with domain of definition $\mathcal{D}(A) = H^2 \cap H_0^1$. It is well known that A is self-adjoint positive definite and that the eigenvalue problem

$$A\phi_j = \mu_j \phi_j$$

provides an orthonormal basis $\{\phi_j\}_{j=1}^{\infty}$ for H and an increasing sequence of eigenvalues

$$(6.2) \qquad 0 < \mu_1 < \mu_2 \le \cdots \le \mu_j \le \cdots, \quad \mu_j \approx j^{2/d} \to \infty \text{ as } j \to \infty.$$

The operator $-A$ is the infinitesimal generator of the semigroup $S(t) = \mathrm{e}^{-tA} \in L(H)$ defined by

$$S(t)v = \mathrm{e}^{-tA} v = \sum_{j=1}^{\infty} \mathrm{e}^{-t\mu_j} \langle v, \phi_j \rangle \phi_j.$$

The semigroup is analytic and, in particular, by a simple calculation using Parseval's identity we have

$$\int_0^T \|A^{1/2}\mathrm{e}^{-tA}v\|^2\,\mathrm{d}t = \int_0^T \sum_j \mu_j \mathrm{e}^{-2t\mu_j}\langle v,\phi_j\rangle^2\,\mathrm{d}t \le \frac{1}{2}\|v\|^2. \tag{6.3}$$

We define norms

$$\|v\|_{\dot H^s} = \Big(\sum_j \mu_j^s \langle v,\phi_j\rangle\Big)^{1/2} = \|A^{s/2}v\|, \quad s\in\mathbf{R}.$$

For $s \ge 0$ we define the corresponding spaces:

$$\dot H^s = \Big\{v\in H : \|v\|_{\dot H^s} < \infty\Big\},$$

$\dot H^{-s}$ is the closure of H with respect to the $\dot H^s$-norm.

The negative order space $\dot H^{-s}$ can be identified with the dual space $(\dot H^s)^*$. Then we have $\dot H^s \subset H = \dot H^0 \subset \dot H^{-s}$. It is known that $\dot H^1 = H_0^1$, $\dot H^2 = H^2 \cap H_0^1 = \mathcal{D}(A)$.

Let $Q \in L(U) = L(H)$, $Q \ge 0$, $U_0 = Q^{1/2}(U) \subset U = H$, and let $\{e_k\}$ be an orthonormal basis in U_0 and $f_k = Q^{-1/2}e_k$, so that $\{f_k\}$ is an orthonormal basis in $U = H$.

There are two possibilities for the choice of $\tilde U$ and J as required in Section 4. The first one is

$$\tilde U = U = H, \quad J : U_0 \to U, \quad J = A^{-s/2}.$$

We must choose s so that J is Hilbert-Schmidt:

$$\|J\|^2_{L_2(U_0,U)} = \|JQ^{1/2}\|^2_{L_2(U)} = \|A^{-s/2}Q^{1/2}\|^2_{L_2(H)} < \infty. \tag{6.4}$$

In order to see what this means, we compute s under some assumptions on Q. If $\mathrm{Tr}(Q) < \infty$, then we may take $s = 0$, because

$$\|Q^{1/2}\|^2_{L_2(H)} = \mathrm{Tr}(Q) < \infty.$$

If $Q = I$, then using (6.2) we obtain

$$\|A^{-s/2}Q^{1/2}\|^2_{L_2(H)} = \|A^{-s/2}\|_{L_2(H)} = \sum_j \|A^{-s/2}\phi_j\|^2$$
$$= \sum_j \mu_j^{-s} \approx \sum_j \left(j^{2/d}\right)^{-s} = \sum_j j^{-2s/d} < \infty,$$

if $-2s/d < -1$, that is $s > \frac{d}{2}$. More generally, if $Q = A^{-\gamma}$, $\gamma \geq 0$, then

$$\|A^{-s/2}Q^{1/2}\|^2_{L_2(H)} = \|A^{-s/2}A^{-\gamma/2}\|^2_{L_2(H)} < \infty, \quad \text{if } s > \frac{d}{2} - \gamma.$$

Now, according to Proposition 4.3, we have that

$$\tilde{W}(t) = \sum_k \beta_k(t) J e_k = \sum_k \beta_k(t) A^{-s/2} e_k, \quad \tilde{W} \in \mathcal{M}^2_T(H),$$

is a $\tilde{Q}$-Wiener process in $H = \tilde{U}$. Moreover,

$$\tilde{Q} = JJ^* = A^{-s} = A^{-s/2}Q^{1/2}(A^{-s/2}Q^{1/2})^*$$

with, according to our assumption (6.4),

$$\mathrm{Tr}(\tilde{Q}) = \|A^{-s/2}Q^{1/2}\|^2_{L_2(H)} < \infty.$$

Now

$$\begin{aligned}
\|\tilde{W}\|^2_{\mathcal{M}^2_T(H)} &= \sup_t \mathbf{E}\big(\|\tilde{W}(t)\|^2\big) \\
&= \sup_t \sum_j \sum_k \underbrace{\mathbf{E}(\beta_k(t)\beta_j(t))}_{=t\delta_{jk}} \langle A^{-s/2}e_k, A^{-s/2}e_j\rangle \\
&= T\sum_k \|A^{-s/2}e_k\|^2 = T\sum_k \|A^{-s/2}Q^{1/2}\underbrace{Q^{-1/2}e_k}_{=f_k}\|^2 \\
&= T\sum_k \|A^{-s/2}Q^{1/2}f_k\|^2 = T\|A^{-s/2}Q^{1/2}\|^2_{L_2(H)} < \infty.
\end{aligned}$$

Note also

$$\tilde{W}(t) = A^{-s/2} \sum_k \beta_k(t) e_k = A^{-s/2} W(t)$$

with $W \in \mathcal{M}_T^2(\dot{H}^{-s})$.

The other choice of $\tilde{U}$ is $\tilde{U} = \dot{H}^{-s}$. Recall that $H = U = \dot{H}^0 \subset \dot{H}^{-s} = \tilde{U}$ for $s \geq 0$, let $J : H \hookrightarrow \dot{H}^{-s}$ be the inclusion, and require that

$$\|J\|^2_{L_2(U_0,\tilde{U})} = \|Q^{1/2}\|^2_{L_2(H,\dot{H}^{-s})} = \|A^{-s/2} Q^{1/2}\|^2_{L_2(H)} < \infty,$$

which is the same condition on s as (6.4). Now

$$\tilde{W}(t) = W(t) = \sum_k \beta_k(t) e_k,$$

with $W \in \mathcal{M}_T^2(\dot{H}^{-s})$. Thus, in both cases we define a (possibly cylindrical) Wiener process $W(t) = \sum_k \beta_k(t) e_k$, where $W \in \mathcal{M}_T^2(\dot{H}^{-s})$, if s satisfies (6.4). The stochastic integral $\int_0^t \Phi \,\mathrm{d}W$ is independent of the choice of J according to Remark 4.8.

The stochastic heat equation (6.1) can now be written

$$\begin{aligned} &\mathrm{d}X + AX\,\mathrm{d}t = \mathrm{d}W, \quad t > 0, \\ &X(0) = 0, \end{aligned} \tag{6.5}$$

where, for simplicity, we have set $X_0 = 0$, $f = 0$. It is of the form (5.1) with $B = I$, and according to Theorems 5.8 and 5.11 its unique weak solution is given by the stochastic convolution

$$X(t) = W_A(t) = \int_0^t S(t-s)\,\mathrm{d}W(s)$$

provided that

$$\int_0^T \|S(t) Q^{1/2}\|^2_{L_2(H)}\,\mathrm{d}t < \infty. \tag{6.6}$$

Using (6.3) and an orthonormal basis $\{f_k\}$ we compute

$$\begin{aligned}\int_0^T \|S(t)Q^{1/2}\|^2_{L_2(H)}\,\mathrm{d}t &= \int_0^T \|\mathrm{e}^{-tA}Q^{1/2}\|^2_{L_2(H)}\,\mathrm{d}t\\ &= \int_0^T \sum_k \|\mathrm{e}^{-tA}Q^{1/2}f_k\|^2\,\mathrm{d}t\\ &= \sum_k \int_0^T \|A^{1/2}\mathrm{e}^{-tA}A^{-1/2}Q^{1/2}f_k\|^2\,\mathrm{d}t\\ &\le \frac{1}{2}\sum_k \|A^{-1/2}Q^{1/2}f_k\|^2 = \frac{1}{2}\|A^{-1/2}Q^{1/2}\|^2_{L_2(H)}.\end{aligned}$$

Thus (6.6) holds if

$$\|A^{-1/2}Q^{1/2}\|_{L_2(H)} < \infty, \tag{6.7}$$

which is (6.4) with $s=0$. Then $W_A \in C\big([0,T],L_2(\Omega,\mathcal{F},P;H)\big)$ according to Theorem 5.5.

More generally, using the isometry (3.2) and (6.3) we compute, for $\beta \ge 0$,

$$\begin{aligned}\mathbf{E}\big(\|W_A(t)\|^2_{\dot H^\beta}\big) = \mathbf{E}\big(\|A^{\beta/2}W_A(t)\|^2\big) &= \mathbf{E}\Big(\Big\|\int_0^t A^{\beta/2}\mathrm{e}^{-(t-s)A}\,\mathrm{d}W(s)\Big\|^2\Big)\\ &= \int_0^t \|A^{\beta/2}\mathrm{e}^{-(t-s)A}Q^{1/2}\|^2_{L_2(H)}\,\mathrm{d}s\\ &= \int_0^t \sum_k \|A^{\beta/2}\mathrm{e}^{-(t-s)A}Q^{1/2}f_k\|^2_H\,\mathrm{d}s\\ &= \sum_k \int_0^t \|A^{1/2}\mathrm{e}^{-sA}A^{(\beta-1)/2}Q^{1/2}f_k\|^2\,\mathrm{d}s\\ &\le \frac{1}{2}\sum_k \|A^{(\beta-1)/2}Q^{1/2}f_k\|^2 = \frac{1}{2}\|A^{(\beta-1)/2}Q^{1/2}\|^2_{L_2(H)}.\end{aligned}$$

So

$$\|W_A(t)\|_{L_2(\Omega,\mathcal{F},P;\dot H^\beta)} \le \|A^{(\beta-1)/2}Q^{1/2}\|_{L_2(H)}, \tag{6.8}$$

provided that

$$\|A^{(\beta-1)/2}Q^{1/2}\|_{L_2(H)} < \infty. \tag{6.9}$$

In particular, if $\mathrm{Tr}(Q) < \infty$, then we may take $\beta = 1$, while if $Q = I$, then we have, by (6.2),

$$\|A^{(\beta-1)/2}\|^2_{L_2(H)} = \sum_k \mu_k^{\beta-1} \approx \sum_k k^{2(\beta-1)/d} < \infty, \tag{6.10}$$

if $2(\beta-1)/d < -1$, that is, we need $0 \le \beta < 1 - d/2$, which only holds if $d = 1$ and $\beta < 1/2$. Thus, for a cylindrical Wiener process ($Q = I$) the solution exists only if $d = 1$. In higher dimensions we need a covariance operator with stronger smoothing effect, for example, if $Q = A^{-\gamma}$ then (6.9) implies $\gamma > \beta - 1 + d/2$.

6.2 The wave equation

We consider the stochastic wave equation

$$\begin{aligned} &\mathrm{d}\dot{u} - \Delta u\,\mathrm{d}t = \mathrm{d}W && \text{in } \mathcal{D} \times \mathbf{R}_+, \\ &u = 0 && \text{on } \partial\mathcal{D} \times \mathbf{R}_+, \\ &u(\cdot,0) = u_0,\ \dot{u}(\cdot,0) = u_1 && \text{in } \mathcal{D}. \end{aligned} \tag{6.11}$$

We let $A = -\Delta$ with $\mathcal{D}(A) = H^2 \cap H^1_0 = \dot{H}^2$, $U = \dot{H}^0 = L_2(\mathcal{D})$ and W be a Q-Wiener process on U as in the previous section. We put

$$X = \begin{bmatrix} u \\ \dot{u} \end{bmatrix}, \quad \xi = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}, \quad H = \dot{H}^0 \times \dot{H}^{-1}.$$

Now we can write

$$
\begin{aligned}
\mathrm{d}X = \begin{bmatrix} \mathrm{d}u \\ \mathrm{d}\dot{u} \end{bmatrix} &= \begin{bmatrix} \dot{u}\,\mathrm{d}t \\ \Delta u\,\mathrm{d}t + \mathrm{d}W \end{bmatrix} \\
&= \begin{bmatrix} X_2 \\ -AX_1 \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W \\
&= \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} X\,\mathrm{d}t + \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W \\
&= \tilde{A}X\,\mathrm{d}t + B\,\mathrm{d}W,
\end{aligned}
$$

where

$$
\tilde{A} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}.
$$

So we have

$$
\begin{aligned}
&\mathrm{d}X = \tilde{A}X\,\mathrm{d}t + B\,\mathrm{d}W, \quad t > 0, \\
&X(0) = \xi,
\end{aligned} \tag{6.12}
$$

where

$$
\mathcal{D}(\tilde{A}) = \left\{ x \in H : \tilde{A}x = \begin{bmatrix} x_2 \\ -Ax_1 \end{bmatrix} \in H = \dot{H}^0 \times \dot{H}^{-1} \right\} = \dot{H}^1 \times \dot{H}^0.
$$

The operator $\tilde{A}$ is the generator of a strongly continuous semigroup $S(t) = \mathrm{e}^{t\tilde{A}}$ on H and $B \in L(U,H)$. Hence, in this case, $U \neq H$ and $B \neq I$. In order to see what S is, we note that $y(t) = S(t)x$ is the solution of

$$
\dot{y} = \tilde{A}y; \quad y(0) = x,
$$

that is,

$$
\ddot{y}_1 + Ay_1 = 0; \quad y_1(0) = x_1,\ \dot{y}_1(0) = x_2.
$$

We solve it using an eigenfunction expansion:

$$\begin{aligned} y_1(t) &= \sum_{j=1}^{\infty} \cos(\sqrt{\mu_j} t)\langle x_1, \phi_j\rangle \phi_j + \frac{1}{\sqrt{\mu_j}} \sin(\sqrt{\mu_j} t)\langle x_2, \phi_j\rangle \phi_j \\ &= \cos(tA^{1/2})x_1 + A^{-1/2}\sin(tA^{1/2})x_2, \end{aligned}$$

and

$$y_2 = \dot{y}_1(t) = -A^{1/2}\sin(tA^{1/2})x_1 + \cos(tA^{1/2})x_2.$$

These are called cosine and sine operator functions. Now we can write the semigroup as

$$S(t) = \mathrm{e}^{t\tilde{A}} = \begin{bmatrix} \cos(tA^{1/2}) & A^{-1/2}\sin(tA^{1/2}) \\ -A^{1/2}\sin(tA^{1/2}) & \cos(tA^{1/2}) \end{bmatrix}.$$

With $\xi = 0$ the evolution problem (6.12) has the unique weak solution

$$\begin{aligned} X(t) = W_{\tilde{A}}(t) &= \int_0^t S(t-s)B\,\mathrm{d}W(s) \\ &= \begin{bmatrix} \int_0^t A^{-1/2}\sin((t-s)A^{1/2})\,\mathrm{d}W(s) \\ \int_0^t \cos((t-s)A^{1/2})\,\mathrm{d}W(s) \end{bmatrix}. \end{aligned}$$

Theorem 5.5 says that $X \in C([0,T]; L_2(\Omega, \mathcal{F}, P; H))$ if

$$\int_0^T \|S(t)BQ^{1/2}\|^2_{L_2(U,H)}\,\mathrm{d}t < \infty.$$

This condition is

$$\begin{aligned} &\int_0^T \|S(t)BQ^{\frac{1}{2}}\|^2_{L_2(U,H)}\,\mathrm{d}t = \int_0^T \sum_k \|S(t)BQ^{\frac{1}{2}} f_k\|^2_H\,\mathrm{d}t \\ &= \int_0^T \sum_k \Big(\|A^{-\frac{1}{2}}\sin(tA^{\frac{1}{2}})Q^{\frac{1}{2}} f_k\|^2_{\dot{H}^0} + \|\cos(tA^{\frac{1}{2}})Q^{\frac{1}{2}} f_k\|^2_{\dot{H}^{-1}}\Big)\,\mathrm{d}t \\ &= \int_0^T \Big(\|A^{-\frac{1}{2}}\sin(tA^{\frac{1}{2}})Q^{\frac{1}{2}}\|^2_{L_2(\dot{H}^0)} + \|A^{-\frac{1}{2}}\cos(tA^{\frac{1}{2}})Q^{\frac{1}{2}}\|^2_{L_2(\dot{H}^0)}\Big)\,\mathrm{d}t. \end{aligned}$$

This must be finite. For example, if $\mathrm{Tr}(Q) < \infty$:

$$\|A^{-\frac{1}{2}}\sin(tA^{\frac{1}{2}})Q^{\frac{1}{2}}\|^2_{L_2(\dot H^0)} \le \|A^{-\frac{1}{2}}\|^2_{L(\dot H^0)}\|\sin(tA^{\frac{1}{2}})\|^2_{L(\dot H^0)}\,\mathrm{Tr}(Q) < \infty,$$

and similarly for cosine, so the condition holds. Here we used

$$\|ST\|_{L_2(\dot H^0)} \le \|S\|_{L(\dot H^0)}\|T\|_{L_2(\dot H^0)},$$

see Remark 1.4. For $Q = I$ we have

$$\begin{aligned}\|A^{-\frac{1}{2}}\sin(tA^{\frac{1}{2}})Q^{\frac{1}{2}}\|^2_{L_2(\dot H^0)} &= \|A^{-\frac{1}{2}}\sin(tA^{\frac{1}{2}})\|^2_{L_2(\dot H^0)}\\ &\le \|A^{-\frac{1}{2}}\|^2_{L_2(\dot H^0)}\|\sin(tA^{\frac{1}{2}})\|^2_{L(\dot H^0)} \le \|A^{-\frac{1}{2}}\|^2_{L_2(\dot H^0)}.\end{aligned}$$

Here $\|A^{-\frac{1}{2}}\|_{L_2(\dot H^0)} < \infty$ if and only if $d = 1$, see (6.10) with $\beta = 0$.

More generally, we compute the norm of order $\beta \ge 0$. For the first component $X_1 = u$ we have:

$$\begin{aligned}\mathbf{E}\big(\|X_1(t)\|_{\dot H^\beta}\big) &= \mathbf{E}\Big(\Big\|\int_0^t A^{\beta/2}A^{-\frac{1}{2}}\sin((t-s)A^{\frac{1}{2}})\,\mathrm{d}W\Big\|^2\Big)\\ &= \int_0^t \|A^{(\beta-1)/2}\sin(sA^{\frac{1}{2}})Q^{\frac{1}{2}}\|^2_{L_2(\dot H^0)}\,\mathrm{d}s\\ &= \int_0^t \|\sin(sA^{\frac{1}{2}})A^{(\beta-1)/2}Q^{\frac{1}{2}}\|^2_{L_2(\dot H^0)}\,\mathrm{d}s\\ &\le \int_0^t \underbrace{\|\sin(sA^{\frac{1}{2}})\|^2_{L(\dot H^0)}}_{\le 1}\,\mathrm{d}s\,\|A^{(\beta-1)/2}Q^{\frac{1}{2}}\|^2_{L_2(\dot H^0)}\,\mathrm{d}s\\ &\le t\|A^{(\beta-1)/2}Q^{\frac{1}{2}}\|^2_{L_2(\dot H^0)}.\end{aligned}$$

So we get the same condition for regularity of order β as for the heat equation, see (6.9). For the second component $X_2 = \dot u$ we obtain similarly

$$\mathbf{E}\big(\|X_2(t)\|^2_{\dot H^{\beta-1}}\big) \le t\|A^{(\beta-1)/2}Q^{\frac{1}{2}}\|^2_{L_2(\dot H^0)}.$$

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