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RESEARCH ARTICLE

A Remark on the Norm of Integer Order Favard Spaces

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Abstract

For a generator A of a C_0-semigroup $T(\cdot)$ on a Banach space X we consider the semi-norm $M_x^k := \limsup_{t\to 0+} \|t^{-1}(T(t)-I)A^{k-1}x\|$ on the Favard space $\mathcal{F}_k$ of order k associated with A. The use of this semi-norm is motivated by the functional analytic treatment of time-discretization methods of linear evolution equations. We show that sharp inequalities for bounded linear operators on $\mathcal{D}(A^k)$ can be extended to the larger space $\mathcal{F}_k$ by using the semi-norm $M_{(\cdot)}^k$. We also show that $M_{(\cdot)}^k$ is a norm equivalent to the norms that are usually considered in the literature if A has a bounded inverse.

1. Introduction

Let X be a Banach space and let A: $X \supset \mathcal{D}(A) \to X$ denote the generator of a C_0-semigroup $T(\cdot)$ of type $(M,0)$; i.e., for some $M \geq 1$ we have $\|T(t)\| \leq M$ for all $t \geq 0$. We denote the algebra of bounded linear operators on X by $\mathcal{B}(X)$. In the functional analytic treatment of time-discretization methods for linear evolution equations, often error estimates of the type

$$\|S_n x - T(t)x\| \leq C f_k(n) \|A^k x\|,\ x \in \mathcal{D}(A^k) \tag{1}$$

are proved, where $k \in \mathbb{N}$, $S_n \in \mathcal{B}(X)$ for all $n \in \mathbb{N}\backslash\{0\}$, $C > 0$ and $f_k(n) \to 0$ as $n \to \infty$ (see, for example, [1], [3], [4, Chapters 3–6], [8], [9], [10, Chapter 3]). The error estimate is sharp if there exists $c > 0$ such that

$$c f_k(n) \leq \sup_{\substack{x\in\mathcal{D}(A^k)\\ \|A^k x\|\leq 1}} \|S_n x - T(t)x\| \leq C f_k(n) \text{ for all } n \in \mathbb{N}\backslash\{0\} \tag{2}$$

(see, for example, [3], [4, Chapters 3–6], [8], [10, Chapter 3]). We briefly outline one possible way to construct such approximating S_n sequences via rational approximations to the exponential function. In numerical analysis this method is often called time discretization. Assume that a rational function r *approximates the exponential function to order* $q \geq 1$; that is,

$$r(z) = e^z + O(z^{q+1}) \text{ as } z \to 0. \tag{3}$$

This means that the Tayor series expansions around the origin of the rational function r and the exponential function $z \mapsto e^z$ coincide up to and including the term containing z^q. Assume further that the function r is A-*stable*;[1] that is, $|r(z)| \le 1$ for $\operatorname{Re} z \le 0$. A commonly used A-stable rational approximation of the exponential function to order 1 is the Backward Euler method $r_{BE}(z) := (1-z)^{-1}$; the most commonly used rational approximation of the exponential function to order 2 is the Crank-Nicolson method $r_{CN}(z) := (2+z)(2-z)^{-1}$. Let us now consider the initial value problem

$$\begin{aligned} \dot{u}(t) &= au(t),\ a \in \mathbb{C},\ t \ge 0, \\ u(0) &= x \in \mathbb{C}, \end{aligned}$$

with solution $u(t) = e^{at}x$. Assume that $|e^{ta}| \le M$ for all $t \ge 0$; that is, $\operatorname{Re} a \le 0$. Let us fix $t > 0$, and let r be an A-stable approximation of the exponential to order q. Then, by the binomial formula, we see that for n large enough

$$\begin{aligned} \left| r^n(\frac{t}{n}a)x - e^{at}x \right| &= \left| \left(r(\frac{t}{n}a)x - e^{\frac{t}{n}a}x \right) \sum_{k=0}^{n-1} r^{n-k}(\frac{t}{n}a) e^{\frac{t}{n}ka} \right| \\ &\le C \left| (\frac{t}{n}a)^{q+1} xn \right| = Ct(\frac{t}{n})^q \left| a^{q+1}x \right|. \end{aligned} \tag{4}$$

Now let X be a Banach space and A: $X \supset \mathcal{D}(A) \to X$ a linear operator generating a C_0-semigroup $T(\cdot)$ of type $(M, 0)$. Then the abstract Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t),\ t \ge 0, \\ u(0) &= x \in X \end{aligned}$$

has a unique solution u given by $u(t) = T(t)x$ where $T(\cdot)$ is the semigroup generated by A. Employing the time-discretization process described in (4) in this abstract framework, we might suspect that for $x \in \mathcal{D}(A^{q+1})$ we have, in analogy to (4),

$$\left\| r^n\left(\frac{t}{n}A\right)x - T(t)x \right\| \le Ct\left(\frac{t}{n}\right)^q \|A^{q+1}x\|. \tag{5}$$

To prove (5), it is tempting to use the binomial formula as in (4) and then move the norm into the summation. It turns out that the latter approach fails because, in general, $\|r^n(\frac{t}{n}A)\|$ is not bounded (see, for example, [3], [9], [10, Chapter 3]). However, for A-stable rational functions and generators of bounded C_0-semigroups (5) is still true (see [3], [9] and [10, Chapter 3]) but the proof requires heavy tools from both classical and functional analysis. It is also

[1] Here the letter A does not refer to the operator A mentioned above.

possible to prove estimates on $\mathcal{D}(A^k)$ for $k = 0, 1, 2, ..., q, k \neq (q+1)/2$, and one obtains estimates of the type (1) with $f_k(n) = (t/n)^{\theta(k)}$ where $\theta(k) = k - 1/2$ if $0 \leq k < (q+1)/2$ and $\theta(k) = kq/(q+1)$ if $(q+1)/2 \leq k \leq q+1$ (see, [3] or [10, Chapter 3]). This means that the order of convergence is optimal on $\mathcal{D}(A^{q+1})$ and decreases if the initial values are less smooth; in particular, on $\mathcal{D}(A)$ the order of convergence is $n^{-1/2}$ and, in general, for arbitrary $x \in X$ the distance between $r^n(\frac{t}{n}A)x$ and $T(t)x$ may increase of order $\sqrt{n}$ as $n \to \infty$. It can also be shown that these estimates cannot be improved in general (see, for example, [2] or [10, Chapter 3]).

The main purpose of this note is to show that error estimates of type (1) and (2) can be extended to Favard spaces $\mathcal{F}_k$ with the same constants c and C and with the same convergence order $f_k(n)$ if one considers the Banach-norm $M_x^k := \limsup_{t\to 0+} \|t^{-1}(T(t) - I)A^{k-1}x\|$ on $\mathcal{F}_k$.

2. Favard spaces

For $\alpha > 0$, $\alpha = l + \beta$, $\beta \in (0, 1]$, $l \in \mathbb{N}$, the space

$$\mathcal{F}_\alpha := \left\{ x \in \mathcal{D}(A^l)\colon \sup_{t>0} \|t^{-\beta}(T(t) - I)A^l x)\| < +\infty \right\} \tag{6}$$

is called the *Favard space of order* α. From the definition it is clear that $\mathcal{D}(A^k) \subset \mathcal{F}_k$ for $k \in \mathbb{N}$. Moreover, if X is reflexive, then $\mathcal{D}(A) = \mathcal{F}_1$ (see, for example, [7, Corollary 5.21]). An easy application of the uniform boundedness principle shows that, for bounded semigroups, the Favard space $\mathcal{F}_k$ ($k \in \mathbb{N}\backslash\{0\}$) consists of those $x \in \mathcal{D}(A^{k-1})$ for which $t \mapsto \langle T(t)A^{k-1}x, x^*\rangle$ is Lipschitz continuous for all $x^* \in X^*$. For $x \in \mathcal{F}_k$, ($k \in \mathbb{N}\backslash\{0\}$) we define

$$M_x^k := \limsup_{t\to 0+} \|t^{-1}(T(t) - I)A^{k-1}x\|. \tag{7}$$

Observe that if $x \in \mathcal{D}(A^k)$, then $M_x^k = \|A^k x\|$. If A^{-1} exists, then $x \mapsto M_x^k$ defines a norm on $\mathcal{F}_k$ and $(\mathcal{F}_k, M_{(\cdot)}^k)$ is a Banach space which is continuously embedded into X (see Proposition 2 below). In case that $\alpha > 0$ is not an integer, we remark that one cannot extend this functional to $\mathcal{F}_\alpha$ to obtain a norm by setting $M_x^\alpha := \limsup_{t\to 0+} \|t^{-\beta}(T(t) - I)A^l x)\|$, $\alpha = l + \beta$, $\beta \in (0, 1)$ and $l = 0, 1, ...$ Indeed, if $\alpha \notin \mathbb{N}$, then $M_x^\alpha = 0$ does not imply that $x = 0$, rather that x belongs to the abstract Hölder space of order α so $x \mapsto M_x^\alpha$ does not define a norm anymore (for an introduction to abstract Favard and Hölder spaces, see, for example, [5], [6], [7], [11]). If X is not reflexive, then in general $\mathcal{D}(A^k)$ is a true subset of $\mathcal{F}_k$. For example, for the right-translation semigroup on $X := C_0(\mathbb{R})$, the Banach space of all continuous functions on $\mathbb{R}$ vanishing at infinity endowed with the supremum norm, the domain $\mathcal{D}(A) = \{f \in C_0(\mathbb{R})\colon f' \in C_0(\mathbb{R})\}$ of $Af := -f'$ is smaller than $\mathcal{F}_1 = \{f \in C_0(\mathbb{R})\colon f \text{ is uniformly Lipschitz continuous on } \mathbb{R}\}$.

Or, if we take the same semigroup on $X = L_1(\mathbb{R})$, then $\mathcal{D}(A) = \{f \in L_1(\mathbb{R})\colon f$ is absolutely continuous on $\mathbb{R}$ and $f' \in L_1(\mathbb{R})\}$, while $\mathcal{F}_1 = \{f \in L_1(\mathbb{R})\colon f$ is of bounded total variation on $\mathbb{R}\}$. As an other example consider a continuous function $q\colon \mathbb{R} \mapsto \mathbb{C}$ and the multiplication operator $(M_q, \mathcal{D}(M_q))$ on $C_0(\mathbb{R})$ defined by $(M_q f)(s) = q(s)f(s)$ on the domain $\mathcal{D}(M_q) := \{f \in C_0(\mathbb{R})\colon fq \in C_0(\mathbb{R})\}$ with q invertible, i.e., $\frac{1}{q} \in C_b(\mathbb{R})$. Here $C_b(\mathbb{R})$ stands for the Banach space of bounded continuous functions on $\mathbb{R}$. It it is shown in [12, Proposition 3.1] that in this case $\mathcal{F}_1 = \{f \in C_0(\mathbb{R})\colon fq \in C_b(\mathbb{R})\}$ which is again larger than $\mathcal{D}(M_q)$. For further concrete examples we refer to [12, Section 3].

The following proposition shows how certain norm estimates on $\mathcal{D}(A^k)$ can be lifted in a simple way to $\mathcal{F}_k$.

Proposition 1. *Assume that A is a generator of a C_0-semigroup $T(\cdot)$ and $S \in \mathcal{B}(X)$. If $0 \le c \le C \le \infty$, then the following are equivalent for all $k \in \mathbb{N}\backslash\{0\}$.*

(i) $c\|A^k x\| \le \|Sx\| \le C\|A^k x\|$ *for all* $x \in \mathcal{D}(A^k)$.

(ii) $cM_x^k \le \|Sx\| \le CM_x^k$ *for all* $x \in \mathcal{F}_k$.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious since $M_x^k = \|A^k x\|$ for $x \in \mathcal{D}(A^k)$. Conversely, assume that (i) holds. Let $x \in \mathcal{F}_k$ and define

$$x_t := \frac{1}{t}\int_0^t T(s)x\,ds,\ t > 0.$$

Then $x_t \in \mathcal{D}(A^k)$ for all $t > 0$ and $x_t \to x$ as $t \to 0+$. Moreover,

$$T(t)A^{k-1}x - A^{k-1}x = A^k \int_0^t T(s)x\,ds,\ t > 0. \tag{8}$$

Hence $\frac{T(t)A^{k-1}x - A^{k-1}x}{t} = A^k x_t,\ t > 0$, and $M_x^k = \limsup_{t\to 0+} \|A^k x_t\|$. Thus, if (i) holds, then

$$c \limsup_{t\to 0+} \|A^k x_t\| \le \limsup_{t\to 0+} \|Sx_t\| \le C \limsup_{t\to 0+} \|A^k x_t\|.$$

Therefore, from the continuity of S it follows that $cM_x^k \le \|Sx\| \le CM_x^k$. ∎

Observe that in Proposition 1 the semigroup does not have to be bounded. In that case the proposition is still true only $M_{(\cdot)}^k$ is a seminorm. Also, as a consequence, sharp estimates of the form (2) extend automaically from $\mathcal{D}(A^k)$ to $\mathcal{F}_k$ one only has to substitute $\mathcal{F}_k$ and M_x^k for $\mathcal{D}(A^k)$ and $\|A^k x\|$ in (2), respectively. Indeed, the upper estiamte follows immediately from Proposition 1 and the lower estimate follows from the facts that $\mathcal{D}(A^k) \subset \mathcal{F}_k$ and that $M_x^k = \|A^k x\|$ if $x \in \mathcal{D}(A^k)$.

In the literature, an integer order Favard space is usually considered with the norm

$$\|x\|_{\mathcal{F}_k} := \|x\| + \|A^{k-1}x\| + \sup_{t\in(0,\infty)} \left\| \frac{T(t)A^{k-1}x - A^{k-1}x}{t} \right\|,$$

or, if A is invertible, with the norm

$$\|x\|_{\mathcal{F}_k} := \sup_{t\in(0,\infty)} \left\| \frac{T(t)A^{k-1}x - A^{k-1}x}{t} \right\|$$

(see [5, Theorem 3.4.10] and [7, Definitions 5.10 and 5.11]). The reasons why we consider the norm $M^k_{(\cdot)}$ are as follows. First,

$$M^k_x \le |||x|||_{\mathcal{F}_k} \le \|x\|_{\mathcal{F}_k}; \tag{9}$$

that is, $M^k_{(\cdot)}$ is the "sharpest" norm of the three. Secondly, for $x \in \mathcal{D}(A^k)$ we have $\|A^k x\| = M^k_x$; that is, we get back the usual norm on $\mathcal{D}(A^k)$. Finally, as Proposition 1 shows, every upper or lower estimate on $\mathcal{D}(A^k)$ automatically extends to $\mathcal{F}_k$. This means that it is enough to prove convergence on $\mathcal{D}(A^k)$, which can be done via functional calculi in abstract numerical analysis (see, for example, [3], [9], [10], [13]), in order to obtain convergence on Favard spaces. We note that the extension of estimates to non-integer order Favard spaces and other intermediate spaces require more serious machinery, namely, interpolation theory (see [10, Chapter 3]).

The next proposition shows that if A is invertible, then $M^k_{(\cdot)}$ makes $\mathcal{F}_k$ into a Banach space.

Proposition 2. *Let A generate a C_0-semigroup $T(\cdot)$ of type $(M, 0)$ and assume that $A^{-1} \in \mathcal{B}(X)$. Then the space $(\mathcal{F}_k, M^k_{(\cdot)})$ ($k \in \mathbb{N}\setminus\{0\}$) is a Banach space and*

$$\mathcal{D}(A^k) \hookrightarrow \mathcal{F}_k \hookrightarrow \mathcal{D}(A^{k-1}) \hookrightarrow X \tag{10}$$

where[2] we endow $\mathcal{D}(A^l)$ with the norm $\|x\|_{\mathcal{D}(A^l)} := \|A^l x\|$, $l \in \mathbb{N}$.

Proof. Clearly, if $x \in \mathcal{F}_k$ and $\lambda \in \mathbb{C}$, then $M^k_{\lambda x} = |\lambda| M^k_x$. Also, if $x, y \in \mathcal{F}_k$, then $M^k_{x+y} \le M^k_x + M^k_y$. If $x \in \mathcal{F}_k$ and $M^k_x = 0$, then $0 = M^k_x = \limsup_{t\to 0+} \|t^{-1}(T(t) - I)A^{k-1}x\|$. Therefore, $\lim_{t\to 0+} \|t^{-1}(T(t) - I)A^{k-1}x)\|$ exists and equals to 0. This implies that $x \in \mathcal{D}(A^k)$ and $\|A^k x\| = 0$. Since A is invertible, it follows that $x = 0$. Hence $M^k_{(\cdot)}$ defines a norm on $\mathcal{F}_k$. The embedding $\mathcal{D}(A^k) \hookrightarrow \mathcal{F}_k$ is clear since for $x \in \mathcal{D}(A^k)$ we have $\|x\|_{\mathcal{D}(A^k)} = M^k_x$. Since $A^{-(k-1)} \in \mathcal{B}(X)$ for all $x \in \mathcal{D}(A^{k-1})$ we have that

$$\|x\| \le \|A^{-(k-1)}\| \, \|A^{k-1}x\| = \|A^{-(k-1)}\| \, \|x\|_{\mathcal{D}(A^{k-1})}. \tag{11}$$

[2] The symbol $\hookrightarrow$ stands for continuous embedding.

This shows that $\mathcal{D}(A^{k-1}) \hookrightarrow X$. To show that $\mathcal{F}_k \hookrightarrow \mathcal{D}(A^{k-1})$ let $x \in \mathcal{F}_k$ and, analogously to (8), we write

$$A^{-1}\frac{T(t)A^{k-1}x - A^{k-1}x}{t} = \frac{1}{t}\int_0^t T(s)A^{k-1}x\,ds.$$

This implies that $\left\|\frac{1}{t}\int_0^t T(s)A^{k-1}x\,ds\right\| \le \|A^{-1}\|\,\left\|\frac{T(t)A^{k-1}x - A^{k-1}x}{t}\right\|$. Hence,

$$\begin{aligned} \|x\|_{\mathcal{D}(A^{k-1})} &= \|A^{k-1}x\| = \lim_{t\to 0+}\left\|\frac{1}{t}\int_0^t T(s)A^{k-1}x\,ds\right\| \\ &\le \|A^{-1}\|\limsup_{t\to 0+}\left\|\frac{T(t)A^{k-1}x - A^{k-1}x}{t}\right\| \\ &= \|A^{-1}\|\,M_x^k. \end{aligned} \tag{12}$$

To show that $\mathcal{F}_k$ is complete, let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{F}_k$. Since $\mathcal{F}_k \hookrightarrow \mathcal{D}(A^{k-1})$ it follows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{D}(A^{k-1})$, too. Since $\mathcal{D}(A^{k-1})$ is complete we may define

$$\mathcal{D}(A^{k-1}) \ni x := \|\cdot\|_{\mathcal{D}(A^{k-1})} - \lim_{n\to\infty} x_n. \tag{13}$$

To see that $x_n \to x$ in $\mathcal{F}_k$ we first prove that if $y \in \mathcal{F}_k$, then

$$\sup_{t\in(0,\infty)}\left\|\frac{T(t)A^{k-1}y - A^{k-1}y}{t}\right\| \le M\,M_y^k. \tag{14}$$

As in the proof of Proposition 1 we define $y_t := \frac{1}{t}\int_0^t T(s)y\,ds,\ t > 0$. We have that $y_t \in \mathcal{D}(A^k)$ and $y_t \to y$ in X. In addition, $y_t \to y$ in $\mathcal{D}(A^{k-1})$ as $t \to 0+$ since

$$\|\cdot\| - \lim_{t\to 0+} A^{k-1}y_t = \|\cdot\| - \lim_{t\to 0+}\frac{1}{t}\int_0^t T(s)A^{k-1}y\,ds = A^{k-1}y.$$

Since $y_t \in \mathcal{D}(A^k)$ it follows that

$$\frac{T(s)A^{k-1}y_t - A^{k-1}y_t}{s} = A\frac{1}{s}\int_0^s T(r)A^{k-1}y_t\,dr = \frac{1}{s}\int_0^s T(r)A^k y_t\,dr.$$

This implies that $\left\|\frac{T(s)A^{k-1}y_t - A^{k-1}y_t}{s}\right\| \le M\|A^k y_t\|$. We have seen in the proof of Proposition 1 that $\limsup_{t\to 0+}\|A^k y_t\| = M_y^k$. Therefore, since $y_t \to y$ in $\mathcal{D}(A^{k-1})$, we have that

$$\begin{aligned} \left\|\frac{T(s)A^{k-1}y - A^{k-1}y}{s}\right\| &= \lim_{t\to 0+}\left\|\frac{T(s)A^{k-1}y_t - A^{k-1}y_t}{s}\right\| \\ &\le M\limsup_{t\to 0+}\|A^k y_t\| = M\,M_y^k, \end{aligned}$$

which proves (14). We show now that x defined in (13) belongs to $\mathcal{F}_k$ and that $x_n \to x$ in $\mathcal{F}_k$. Indeed, by (13) and (14), we have

$$\begin{aligned}\left\|\frac{T(t)A^{k-1}x - A^{k-1}x}{t}\right\| &= \lim_{n\to\infty}\left\|\frac{T(t)A^{k-1}x_n - A^{k-1}x_n}{t}\right\| \\ &\le \limsup_{n\to\infty}\sup_{t\in(0,\infty)}\left\|\frac{T(t)A^{k-1}x_n - A^{k-1}x_n}{t}\right\| \\ &\le \limsup_{n\to\infty} MM^k_{x_n} < \infty.\end{aligned}$$

This proves that $x \in \mathcal{F}_k$ and that $M^k_x \le M \limsup_{n\to\infty} M^k_{x_n}$. Similarly, we obtain that $M^k_{x-x_n} \le M \limsup_{m\to\infty} M^k_{x_m-x_n}$ and hence

$$\lim_{n\to\infty} M^k_{x-x_n} \le M \lim_{n\to\infty}\limsup_{m\to\infty} M^k_{x_m-x_n} = 0.$$

This implies that $x_n \to x$ in $\mathcal{F}_k$ and the proof is complete. ∎

The next corollary shows that the norms $M^k_{(\cdot)}$, $|||\cdot|||_{\mathcal{F}_k}$ and $\|\cdot\|_{\mathcal{F}_k}$ are equivalent and the proof of Proposition 2 allows us to determine the constants when comparing these norms.[3]

Corollary 3. *Let A generate a C_0-semigroup $T(\cdot)$ of type $(M,0)$ and assume that $A^{-1} \in \mathcal{B}(X)$. Then $M^k_{(\cdot)}$ defines a norm equivalent to $\|\cdot\|_{\mathcal{F}_k}$ and $|||\cdot|||_{\mathcal{F}_k}$ on $\mathcal{F}_k$. Moreover, for $x \in \mathcal{F}_k$,*

$$\frac{1}{M}\,|||x|||_{\mathcal{F}_k} \le \; M^k_x \; \le |||x|||_{\mathcal{F}_k}, \tag{15}$$

$$\frac{1}{M + (1+\|A^{-(k-1)}\|)\|A^{-1}\|}\,\|x\|_{\mathcal{F}_k} \le \; M^k_x \; \le \|x\|_{\mathcal{F}_k}. \tag{16}$$

Proof. The inequalities in (15) follow from (9) and (14) and the inequalities in (16) follow from (9), (11), (12) and (14). ∎

Finally, we mention one possible application. Let $X := L_1(\mathbb{R})$ and $Af := f'$ with maximal domain; that is, $\mathcal{D}(A) := W_{1,1}(\mathbb{R})$ the Sobolev space of order 1. In [2] it is shown (see also, [3] and [10, Chapter 3]) that if we choose the rational function r_{CN} mentioned in the introduction, then for all $f \in W_{1,1}(\mathbb{R})$ we have

$$c_t n^{-\frac{1}{2}} \le \sup_{\substack{f\in W_{1,1}(\mathbb{R})\\ \|f\|_{W_{1,1}(\mathbb{R})}\le 1}} \|r^n_{CN}(\frac{t}{n}A)f - T(t)f\|_{L_1(\mathbb{R})} \le C_t n^{-\frac{1}{2}},\; n \in \mathbb{N}\backslash\{0\}.$$

[3] It is shown in [12, Proposition 1.6] that $M^1_{(\cdot)}$ is equivalent to the the norm $|x| := \inf\{c \in \mathbb{R}\colon \exists (x_n) \in \mathcal{D}(A)$ such that $x = \|\cdot\|-\lim_{n\to\infty} x_n$ and $\sup_n \|Ax_n\| \le c\}$ (see also [7, Exercise 5.23]).

By Proposition 1 and the short discussion thereafter this sharp estimate holds for all $f \in \mathcal{F}_1 = \{f \in L_1(\mathbb{R})\colon\ f$ is uniformly Lipschitz continuous on $\mathbb{R}\}$ with the Sobolev norm replaced by the appropriate Lipschitz norm of f on $\mathbb{R}$.

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