Wave attenuation through multiple rows of scatterers with differing periodicities

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The marginal ice zone
Scattering by a single body

\[ \nabla^2 \phi + k^2 \phi = 0 \]

\[ \phi_{\text{inc}} = e^{ik(x \cos \theta_{\text{inc}} + y \sin \theta_{\text{inc}})} \]
Multiple bodies

\[ (\phi_{inc}, x, y) \]
A periodic row of bodies
A periodic row of bodies

\[ y \]

\[ x \]

\[ \phi_{inc} \]

Transmitted waves
Scattering matrices

- The periodicity $p$ is given by a relationship between the wavenumber $k$ and the spacing $s$.

- The scattering angles $\theta_i$ are defined by the incident angle and $p$.

- The incident angle is always a scattered angle and multiple angles are generated only if $p < 2$.

- Reflection and transmission between the scattering angles are described by $r_{i,j}^{(\pm)}$ and $t_{i,j}^{(\pm)}$ resp.

- Collect these into scattering matrices $R^{(\pm)}$ and $T^{(\pm)}$ resp.
A semi-infinite row of bodies
A periodic row of clusters of bodies
A multiple-row array
A multiple-row array

Transmitted waves
Solution through iteration

- Denote the (known) scattering matrices (i.e. reflection and transmission coefficients) for the individual rows by

\[ R_i^{(\pm)} \text{ and } T_i^{(\pm)}, \]

for \( i = 1, \ldots, M \) (\( M \) is the number of rows in the array).

- Apply a wide spacing approximation (for efficiency).

- Calculate the scattering matrices for rows 1 & 2 from

\[ T_{1,2}^{(-)} = T_1^{(+)} D \left( I - R_2^{(-)} D R_1^{(+)} D \right)^{-1} T_2^{(+)}, \text{ etc.} \]

where \( D = \text{diag. matrix of phase changes between rows.} \)

- Iterate to find the scattering matrices for the entire array

\[ R_{1,M}^{(\pm)} \text{ and } T_{1,M}^{(\pm)}. \]
Coherence problems

![Graph showing coherence problems with transmission on the y-axis and wavenumber on the x-axis.](Image)
Attenuation and averaging

- Want to extract a (dimensionless) attenuation coefficient $\alpha$:

$$ E \approx e^{-\alpha m} $$

where

- $E$ is transmitted energy
- $m$ is number of rows

- Ensemble average over different simulations

$$ E = E(\Lambda) = \sum_{j=1}^{N} \frac{E_j(m)}{N} \quad N, M \text{ large} \left( \sim O(10^2) \right) $$

and ‘fit’ an exponential curve to this.
Effects of periodicity

8 s

12 s

16 s

\( \alpha \) vs. concentration

Concentration ranges from 0 to 0.8 with increments of 0.1.
Effects of periodicity

8 s

12 s

16 s

\[ \alpha \]

\[ \times 10^{-3} \]

\[ \times 10^{-4} \]

\[ \times 10^{-5} \]

concentration
Varying periodicities
Problems with interaction theory
Problems with interaction theory
Problems with interaction theory
Problems with interaction theory
Problems with interaction theory

\[ \phi_{inc} \]

\[ \theta_{inc} \]

\[ \begin{align*}
\theta_1 & \rightarrow \theta_{1,1} & \theta_{1,1,1} \\
\theta_1 & \rightarrow \theta_{1,1} & \theta_{1,1,2} \\
\theta_2 & \rightarrow \theta_{2,1} & \theta_{2,1,1} \\
\theta_2 & \rightarrow \theta_{2,1} & \theta_{2,1,2} \\
\theta_2 & \rightarrow \theta_{2,1} & \theta_{2,2,1} \\
\theta_2 & \rightarrow \theta_{2,1} & \theta_{2,2,2} \\
\theta_1 & \rightarrow \theta_{1,2} & \theta_{1,2,1} \\
\theta_1 & \rightarrow \theta_{1,2} & \theta_{1,2,2} \\
\theta_2 & \rightarrow \theta_{2,2} & \theta_{2,2,1} \\
\theta_2 & \rightarrow \theta_{2,2} & \theta_{2,2,2}
\end{align*} \]
Discretization

\[ y \]

\[ \phi_{inc} \]

\[ \Delta y \]
Approximate

\[ \phi_{inc} \]

\[ \Delta y \]
The discrete system has an underlying periodicity
\[ \tilde{p} = \tilde{p}(\Delta y). \]

It can therefore be solved using the same method as the fixed periodicity problem earlier.

As \( \Delta y \to 0 \) the geometry converges to its intended state.
Simple structures

\[ p = \frac{4}{3} \]

\[ p = \frac{5}{3} \]

Transmission

\[ \cos(\theta) \]

\[ \cos(\theta) \]

\[ \times \text{exact} \]

- 21 points
- 41 points
- 81 points
- 161 points
Convergence (discretization)

Transmission

\[ p \approx 2.5 \]

\[ \cos(\theta) \]

\[ p \approx 1.75 \]

\[ \cos(\theta) \]

\[ p \approx 1 \]

\[ \cos(\theta) \]
Adding periodicity variation \( (p \approx 1.75) \)

\[
\sigma(p) = 0
\]

\[
\sigma(p) = 0.1
\]

\[
\sigma(p) = 0.2
\]

\[
\sigma(p) = 0.4
\]