PROPERTIES OF CONTINUOUS TIME RANDOM WALKS WITH WAITING TIMES

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Abstract. Previous work showed how moving particles that rest along their trajectory lead to time-nonlocal advection-dispersion equations. If the waiting times have infinite mean, the model equation contains a fractional time derivative of order between 0 and 1, for finite mean but infinite variance the model contains a derivative of order between 1 and 2. Finite mean and finite variance lead to a second order in time partial differential equation. In this article we investigate the various solutions with regard to moment growth and scaling properties and show that infinite mean waiting times do not necessarily induce sub-diffusion but can even lead to super-diffusion.

1. Introduction

Continuous time random walks (CTRW) can be used to derive governing equations for anomalous diffusion [12, 13, 14]. The CTRW is a stochastic process model for the movement of an individual particle. In the long-time limit, the process converges to a simpler form whose probability densities solve the governing equation, leading to a useful model for anomalous diffusion. For a simple random walk with mean zero, finite variance particle jumps, the limit process is a Brownian motion $A(t)$ governed by the classical diffusion equation $\partial p/\partial t = C\partial^2 p/\partial x^2$ where $p(x, t)$ is the probability density of the random variable $A(t)$. For symmetric infinite variance jumps (whose probability distribution is assumed regularly varying [11] with some index $0 < \alpha \leq 2$), the limit process $A(t)$ is an α-stable Lévy motion, and the governing equation becomes $\partial p/\partial t = C\partial^\alpha p/\partial|x|^\alpha$ [9]. When waiting times are introduced, the process is altered via subordination. The resulting process is $A(E(t))$ where the process $E(t)$ counts the number of particle jumps by time $t \geq 0$, accounting for the waiting time between particle jumps. In the limit the process $E(t)$ is the inverse or hitting time process for the γ-stable subordinator. The governing equation becomes $\partial^\gamma p/\partial t^\gamma = C\partial^\alpha p/\partial|x|^\alpha$ [12, 13] in the case of $0 < \gamma < 1$ and $-a\partial^\gamma p/\partial t^\gamma + \partial p/\partial t = C\partial^\alpha p/\partial|x|^\alpha + \delta(t)g(x)$ for $1 < \gamma \leq 2$ [3]. In this paper, we investigate properties of the solutions to these equations.

2. The model

In the usual CTRW formalism, the long-time limit for the waiting time process is a γ-stable subordinator $D(t)$ [12]. Then the inverse Lévy process $E(t) = \inf\{x : D(x) > t\}$ counts the number of particle jumps by time $t \geq 0$, reflecting the fact that the time T_n of the nth particle jump and the number $N_t = \max\{n : T_n \leq t\}$ of jumps by time t are

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also inverse processes. When the waiting times between particle jumps have heavy tails with $0 < \gamma < 1$, subordination of the particle location process $A(t)$ via the inverse Lévy process $E(t)$ is necessary in the long-time limit to account for this, which leads to a time derivative of order γ in the governing equation [13]. The process $E(t)$ has the following density representation:

$$p(t,s) := \frac{t}{\gamma s (as)^{1/\gamma}} g_\gamma\left(\frac{t}{(as)^{1/\gamma}}\right), \tag{2.1}$$

where g_γ is the density of the γ-stable subordinator; i.e. has Laplace transform

$$\int_0^\infty e^{-\lambda t} g_\gamma(t) dt = e^{-\lambda^\gamma}.$$

When waiting times have heavy tails of order $1 < \gamma \le 2$, meaning that the probability of waiting longer than t falls off like $t^{-\gamma}$, a different model is needed [6]. In this case, convergence of the waiting time process requires centering to the mean waiting time w, which is not necessary when $0 < \gamma < 1$. Accounting for this leads to a waiting time process $W(t) = D(t) + wt$ where $D(t)$ is a completely positively skewed[1] stable Lévy process with index γ, so that $W(t)$ is a Lévy process with drift (The drift accounts for the mean waiting time). Since $\gamma > 1$, the process $W(t)$ is not strictly increasing, so we use $\mathrm{Max}(t) = \sup\{W(u) : 0 \le u \le t\}$ to represent the particle jump times. Then the inverse or hitting time process $H(t) = \inf\{x : \mathrm{Max}(x) \ge t\}$ counts the number of particle jumps by time t. In [3] we showed that the distribution of the hitting time process with mean waiting time $w = 1$ has the following Laplace - Laplace transform:

$$\int_0^\infty \int_0^\infty e^{-us-\lambda T} H(s,T)\, ds\, dT = \frac{1 - a\lambda^{\gamma-1} + u/q_a(u)}{u(u+\lambda - a\lambda^\gamma)}, \tag{2.2}$$

where q_a is the analytic function which satisfies

$$a q_a(\lambda)^\gamma - q_a(\lambda) = \lambda$$

for all λ in a region containing the right half plane. Furthermore, we inverted (2.2) in [3], which, using scaling, reads for general w,

$$\Pr\{H(t) \le s\} = \int_{\frac{t/w-s}{(as)^{1/\gamma}}}^\infty g_\gamma(u)\, du + \int_0^s \frac{m_a(s-u)}{(au)^{1/\gamma}} g_\gamma\left(\frac{t/w - u}{(au)^{1/\gamma}}\right) du, \tag{2.3}$$

where g_γ is the density of the γ-stable subordinator; i.e. has Fourier transform

$$\int e^{-ikt} g_\gamma(t)\, dt = e^{(ik)^\gamma}$$

and the function m_a satisfies $\int_0^\infty e^{-\lambda t} m_a(t)\, dt = 1/q_a(\lambda)$. The second term in (2.3) is a correction term compensating for the fact that the limiting waiting time process is not monotone. It converges to zero as t increases.

[1] The skew is irrelevant in the normal case $\gamma = 2$.

3. Dimensional Analysis

In order to effectively investigate and compare the hitting time densities we bring them into dimensionless form. In (2.3) we have an expression with four variables, $t, w, a,$ and s. The dimensions are $[T]$*time* for t, $[B]$*bulk jumps* for s, the mean waiting time w has dimension of $[T/B]$. The variable a has dimension $[\tilde{T}^\gamma/B]$, with $\tilde{T}$ being time in relation to the average waiting time. This yields that a has a dimension of $[\tilde{T}^\gamma/B] = [(T/(T/B))^\gamma/B] = [B^{\gamma-1}]$.

In order to non-dimensionalise (2.3) we need to obtain a scaling property for the function m_a. Now, $m_a(ct)$ has the following Laplace transform:

$$\int_0^\infty e^{-\lambda t} m_a(ct)\,dt = \frac{1}{c}\int_0^\infty e^{-\lambda t/c} m_a(t)\,dt = \frac{1}{c}\frac{1}{q_a(\lambda/c)}.$$

Since

$$aq_a(\lambda/c)^\gamma - q_a(\lambda/c) = \lambda/c,$$

we obtain that

$$\frac{a}{c^{\gamma-1}}\left(cq_a(\lambda/c)\right)^\gamma - cq_a(\lambda/c) = \lambda.$$

Hence $cq_a(\lambda/c) = q_{a/c^{\gamma-1}}(\lambda)$ and thus

$$\int_0^\infty e^{-\lambda t} m_a(ct)\,dt = \frac{1}{q_{a/c^{\gamma-1}}(\lambda)} = \int_0^\infty e^{-\lambda t} m_{a/c^{\gamma-1}}(t)\,dt.$$

Therefore

$$m_a(ct) = m_{a/c^{\gamma-1}}(t).$$

Our goal is to find a parameterization of (2.3) such that the parameters are dimensionless and the number of parameters is reduced. Introducing

$$\xi := w/(t/s) = ws/t \text{ and } \rho := a/(t/w)^{\gamma-1}$$

we obtain that

(3.1)

$$\begin{aligned}
\psi(t,s,a,w) = \Pr\{H(t)\le s\} &= \int_{\frac{t/w-s}{(as)^{1/\gamma}}}^\infty g_\gamma(u)\,du + \int_0^s \frac{m_a(s-u)}{(au)^{1/\gamma}} g_\gamma\left(\frac{t/w-u}{(au)^{1/\gamma}}\right)du \\
&= \int_{\frac{1-\xi}{(\rho\xi)^{1/\gamma}}}^\infty g_\gamma(u)\,du + \int_0^{\xi t/w} \frac{m_{\rho(t/w)^{\gamma-1}}(\xi t/w-u)}{(\rho(t/w)^{\gamma-1}u)^{1/\gamma}} g_\gamma\left(\frac{t/w-u}{(\rho(t/w)^{\gamma-1}u)^{1/\gamma}}\right)du \\
&= \int_{\frac{1-\xi}{(\rho\xi)^{1/\gamma}}}^\infty g_\gamma(u)\,du + \int_0^{\xi} \frac{m_\rho(\xi-u)}{(\rho u)^{1/\gamma}} g_\gamma\left(\frac{1-u}{(\rho u)^{1/\gamma}}\right)du \\
&= \psi(\xi,\rho).
\end{aligned}$$

Clearly, the variables ρ and ξ are dimensionless.

4. Plots of First Arrival Densities

In this section we give plots of the first arrival densities of positively skewed Lévy motions (with drift if $\gamma > 1$).

4.1. **The case $\gamma < 1$.** The cumulative distribution function for the first arrival density is given by

$$\Pr\{H(t) \leq s\} = \int_{\frac{t}{(as)^{1/\gamma}}}^{\infty} g_\gamma(u)\, du$$

as can easily be deduced from (2.1). Using $\xi = as/t^\gamma$ we obtain its density in dimensionless form

$$p(\xi) = \frac{1}{\gamma \xi^{1+1/\gamma}} g_\gamma\left(\frac{1}{\xi^{1/\gamma}}\right).$$

The plots of the densities are displayed in Figure 1.

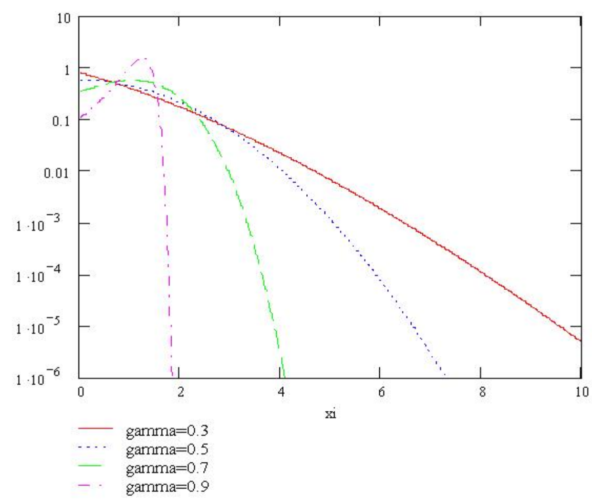

Figure 1. First arrival densities of γ-stables, $\gamma < 1$.

4.2. **The case $1 < \gamma < 2$.** In Figure 2 we show the evolution of the arrival densities for various γ and various "dispersivities", ρ. They were computed using a Fast Fourier Inverter for the Laplace transform. The corresponding Mathcad sheet can be downloaded from `http://www.maths.otago.ac.nz/~bbaeumer`. Note that ρ is proportional to $t^{1-\gamma}$. The density therefore keeps the same shape for longer the smaller γ. In the regime of $\rho \approx 1$, a significant part of the limiting waiting time distribution is still negative, accounting for the possibility of lots of jumps in a short amount of time. This amplifies the fact that the limiting density is not an early time approximation.

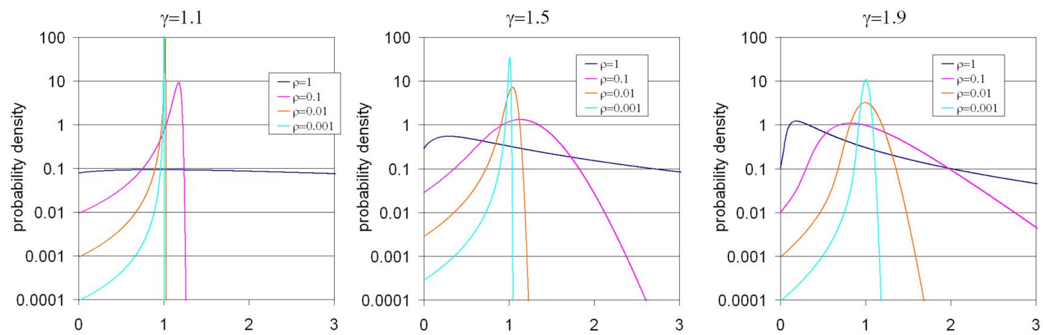

FIGURE 2. First arrival densities of γ-stables, $1 < \gamma < 2$.

4.3. **The case $\gamma = 2$.** In the special case of finite second moments in the waiting time distribution, i.e., $\gamma = 2$, we can explicitly compute the hitting time density of the limit distribution. In this case, (2.2) reduces to

$$\begin{aligned}\int_0^\infty \int_0^\infty e^{-us-\lambda T} dH(s,T)\, dT =& \frac{1-a\lambda+2au/(1+\sqrt{1+4au})}{u+\lambda-a\lambda^2}\\ =& \frac{1-a\lambda-(1-\sqrt{1+4au})/2}{u+\lambda-a\lambda^2}\\ =& \frac{-2a\lambda+1+\sqrt{1+4au}}{-2a(\lambda-(1-\sqrt{1+4au})/2a)(\lambda-(1+\sqrt{1+4au})/2a)}\\ =& \frac{1}{\lambda-(1-\sqrt{1+4au})/2a}.\end{aligned}$$

Inverting in λ yields

$$\int_0^\infty e^{-us} dH(s,T) = \exp(T(1-\sqrt{1+4au})/2a) = \exp(T/2a)\exp\left(-\frac{T}{\sqrt{a}}\sqrt{u+\frac{1}{4a}}\right).$$

Thus, using that the inverse Laplace transform of $\exp(-au^{1/2})$ is $\frac{a}{2\sqrt{\pi}s^{3/2}}\exp(-a^2/4s)$, we obtain

$$\begin{aligned}h(s,t) &:= \partial H(s,t)/\partial s\\ &= \exp\left(t/2a - \frac{s}{4a}\right)\frac{t}{2s\sqrt{\pi a s}}\exp\left(-\frac{t^2}{4as}\right) = \frac{t}{2s\sqrt{\pi a s}}\exp\left(-\frac{(t-s)^2}{4as}\right).\end{aligned} \tag{4.1}$$

In terms of ξ and ρ (4.1) becomes

$$\bar{h}(\xi,\rho) := \frac{\partial \bar{H}(\xi,\rho)}{\partial \xi} = \frac{\partial H(\xi a/\rho, a/\rho)}{\partial \xi} = \frac{1}{2\xi\sqrt{\pi\rho\xi}}\exp\left(-\frac{(1-\xi)^2}{4\rho\xi}\right).$$

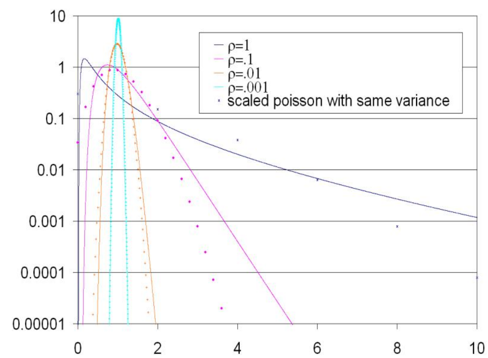

FIGURE 3. Poisson vs First arrival of Gaussian

The canonical choice for a waiting time distribution with finite variance is the exponential distribution. Its hitting time process is the Poisson process. In Figure 3 we plotted the scale-adjusted Poisson distribution against the hitting time density of the Gaussian for various ρ. For large ρ (early time) there is an obvious difference; the Gaussian allows for much further travel than the exponential as the probability for having several jumps in a small amount of time is larger for the Gaussian than for the exponential distribution. For smaller ρ the two distributions converge towards the δ -function at one.

This highlights that we have to treat our densities with caution in the early time regime. This corresponds to avoiding a limiting waiting time distribution with a significant negative part. As a rule of thumb, ρ should be less than 1/10.

5. The first and second moments

In this section we compute the first and second moments of the hitting time densities. We show that for $\gamma < 1$ the first moment grows like t^γ, while the variance grows like $t^{2\gamma}$. For $\gamma > 1$ we show that for large time the first moment grows basically linear in time, while the variance grows like $t^{3-\gamma}$. This will imply that having heavy-tailed waiting times does not necessarily lead to a sub-diffusive process as we have a smearing out effect on the traveling plume which induces the variance to grow faster than linear. The process will only be subdiffusive if $\gamma < 0.5$ or if we have pure diffusion and no advection.

5.1. **The case $\gamma < 1$.** In [4] and [13] we showed that $p(t,s)$ of equation (2.1) has Laplace-Laplace transform

$$\tilde{\tilde{p}}(\lambda,u) := \int_0^\infty \int_0^\infty e^{-us} e^{-\lambda t} p(t,s)\,ds\,dt = \frac{\lambda^{\gamma-1}}{u+\lambda^\gamma}.$$

In order to compute the first moment $\mu(t) = \int_0^\infty sp(t,s)\,ds$, note that

$$\mu(t) = -\frac{d}{du}\int_0^\infty e^{-us}p(t,s)\,ds\bigg|_{u=0}.$$

Hence,

$$\tilde{\mu}(\lambda) = \int_0^\infty e^{-\lambda t}\mu(t)\,dt = -\frac{d}{du}\frac{\lambda^{\gamma-1}}{u+\lambda^\gamma}\bigg|_{u=0} = \lambda^{-1-\gamma}.$$

Thus, for $\gamma < 1$,

$$\mu(t) = t^\gamma/\Gamma(1+\gamma). \tag{5.1}$$

Similarly, the second moment is computed by

$$\tilde{\mu}_2(\lambda) = \frac{d^2}{du^2}\frac{\lambda^{\gamma-1}}{u+\lambda^\gamma}\bigg|_{u=0} = 2\lambda^{-1-2\gamma}.$$

Therefore, for $\gamma < 1$,

$$\sigma^2(t) = \mu_2(t) - \mu^2(t) = t^{2\gamma}(2/\Gamma(1+2\gamma) - 1/(\Gamma(1+\gamma)^2)). \tag{5.2}$$

5.2. **The case** $1 < \gamma < 2$**.** We use the same technique to compute the first and second moments of the first arrival processes for $\gamma > 1$. Recall the the Laplace-Laplace transform of the Hitting time density is given by (2.2). Thus

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\mu(t)dt &:= \int_0^\infty e^{-\lambda t}\int_0^\infty s\,dH(s,t)dt = -\frac{d}{du}\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-us}\,dH(s,t)dt\bigg|_{u=0} \\ &= -\frac{d}{du}\frac{1-a\lambda^{\gamma-1}+u/q_a(u)}{u+\lambda-a\lambda^\gamma}\bigg|_{u=0} = \frac{1-a\lambda^{\gamma-1}}{(\lambda-a\lambda^\gamma)^2} - \frac{1/q_a(0)}{\lambda-a\lambda^\gamma} \\ &= \frac{1-\lambda/q_a(0)}{\lambda^2(1-a\lambda^{\gamma-1})} = \frac{1-\lambda/q_a(0)}{a\lambda^{\gamma+1}(\frac{1}{a\lambda^{\gamma-1}}-1)} = -\frac{1-\lambda/q_a(0)}{a\lambda^{\gamma+1}}\sum_{n=0}^\infty(1/a\lambda^{\gamma-1})^n\end{aligned} \tag{5.3}$$

This inverts to

$$\mu(t) = \sum_{n=0}^\infty (t^{\gamma-1}/a)^n\left(\frac{-t^\gamma}{a\Gamma(1+\gamma+n(\gamma-1))} + \frac{t^{\gamma-1}}{aq_a(0)\Gamma(\gamma+n(\gamma-1))}\right).$$

Unfortunately, this series does not reveal the type of growth of $\mu(t)$ for large t. The following asymptotic expansion does the trick.

Let N be such that $1 < (\gamma-1)(N+1) \leq \gamma$. Then (5.3) equals

$$\frac{1-\lambda/q_a(0)}{\lambda^2(1-a\lambda^{\gamma-1})} = \frac{(a\lambda^{\gamma-1})^{N+1}-\lambda/q_a(0)}{\lambda^2(1-a\lambda^{\gamma-1})} + \frac{1}{\lambda^2}\sum_{n=0}^N(a\lambda^{\gamma-1})^n =: r1(\lambda)+r2(\lambda).$$

A Tauberian theorem for Laplace transforms (see, for example, [2] Thm. 2.6.4) tells us that the behavior of $\lambda r1(\lambda)$ at zero corresponds to the behavior of its inverse at infinity, as long as $\lambda r1(\lambda)$ is bounded and analytic in a sectorial region containing the right half plane.

Hence, the inverse of $r1$ for large values converges to $-1/q(0)$. The second term inverts to $t\sum_{n=0}^{N}\frac{(at^{1-\gamma})^n}{\Gamma(2+n(1-\gamma))}$. Thus

$$\lim_{t\to\infty}\mu(t)-\left(t\sum_{n=0}^{N}\frac{(at^{1-\gamma})^n}{\Gamma(2+n(1-\gamma))}-1/q(0)\right)=0 \tag{5.4}$$

or $\mu(t)\sim t+ct^{2-\gamma}+\ldots$.

We can do the same thing for the second moment. In order to estimate the variance, however, we need an estimate of $\mu(t)^2$. This requires a better estimate of $\mu(t)$; namely, we need an approximating function f such that

$$\mu(t)^2-f(t)^2=(\mu(t)+f(t))(\mu(t)-f(t))\approx ct(\mu(t)-f(t))\to 0.$$

Let M be such that $2<(\gamma-1)(M+1)\le 1+\gamma$ and consider that

$$\frac{1-\lambda/q(0)}{\lambda^2(1-a\lambda^{\gamma-1})}-\left(\frac{1}{\lambda^2}\sum_{n=0}^{M}(a\lambda^{\gamma-1})^n-\frac{1}{q(0)\lambda}\sum_{n=0}^{N}(a\lambda^{\gamma-1})^n\right)=\frac{(a\lambda^{\gamma-1})^{M+1}-(a\lambda^{\gamma-1})^{N+1}\lambda/q(0)}{\lambda^2(1-a\lambda^{\gamma-1})}.$$

Recall that the derivative in λ of a Laplace transform of a function $f(t)$ is the Laplace transform of $-tf(t)$. The derivative of above expression is again bounded in a sectorial region and the limit as $\lambda\to 0$ of λ times the derivative is zero. Hence

$$\lim_{t\to\infty}t\left(\mu(t)-\left(t\sum_{n=0}^{M}\frac{(at^{1-\gamma})^n}{\Gamma(2+n(1-\gamma))}-\frac{1}{q(0)}\sum_{n=0}^{N}\frac{(at^{1-\gamma})^n}{\Gamma(1+n(1-\gamma))}\right)\right)=0.$$

The second moment is computed via

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\int_0^\infty s^2\,dH(s,t)dt &= \frac{d^2}{du^2}\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-us}\,dH(s,t)dt\Big|_{u=0}\\
&=\frac{d^2}{du^2}\frac{1-a\lambda^{\gamma-1}+u/q(u)}{u+\lambda-a\lambda^\gamma}\Big|_{u=0}\\
&=\frac{2\left(1-\lambda/q(0)+\lambda^2\frac{1-a\lambda^{\gamma-1}}{q(0)^2(1-\gamma)}\right)}{\lambda^3(1-a\lambda^{\gamma-1})^2}\\
&=\frac{2(M+1)(a\lambda^{\gamma-1})^{M+1}-2\lambda/q(0)(N+1)(a\lambda^{\gamma-1})^{N+1}}{\lambda^3(1-a\lambda^{\gamma-1})^2}\\
&\quad+\frac{2}{\lambda(1-a\lambda^{\gamma-1})q(0)^2(1-\gamma)}\\
&\quad+\frac{2}{\lambda^3}\sum_{n=0}^{M}(n+1)(a\lambda^{\gamma-1})^n-\frac{2}{q(0)\lambda^2}\sum_{n=0}^{N}(n+1)(a\lambda^{\gamma-1})^n\end{aligned}\tag{5.5}$$

Again, the first two terms are analytic in a sectorial region and its product with λ is bounded. So for large t, the second moment behaves like

$$\int_0^\infty s^2 dH(s,t) \approx 2t^2 \sum_{n=0}^{M} \frac{n+1}{\Gamma(3+n(1-\gamma))}(at^{1-\gamma})^n - \frac{2t}{q(0)} \sum_{n=0}^{N} \frac{n+1}{\Gamma(2+n(1-\gamma))}(at^{1-\gamma})^n + \frac{2}{q(0)^2(1-\gamma)}. \tag{5.6}$$

This gives us an estimate for the variance for large t:

$$\begin{aligned}\sigma^2(t) \approx& 2t^2 \sum_{n=0}^{M} \frac{n+1}{\Gamma(3+n(1-\gamma))}(at^{1-\gamma})^n - \frac{2t}{q(0)} \sum_{n=0}^{N} \frac{n+1}{\Gamma(2+n(1-\gamma))}(at^{1-\gamma})^n + c \\ &- \left(t \sum_{n=0}^{M} \frac{(at^{1-\gamma})^n}{\Gamma(2+n(1-\gamma))} - \frac{1}{q(0)} \sum_{n=0}^{N} \frac{(at^{1-\gamma})^n}{\Gamma(1+n(1-\gamma))} \right)^2 \\ \approx& c_1 t^{3-\gamma} + c_2 t^{4-2\gamma} + \ldots - d_1 t^{2-\gamma} - \ldots + e\end{aligned} \tag{5.7}$$

for some constants c_i, d_i, and e.

5.3. **The case $\gamma = 2$.** Here the previous calculations are much simplified. Recall that $q_a(0)$ satisfies $aq_a(0)^\gamma - q_a(0) = 0$, or in this case, $q_a(0) = 1/a$. Using equation (5.3) we obtain that

$$\tilde{\mu}(\lambda) = \frac{1-a\lambda}{\lambda^2(1-a\lambda)} = 1/\lambda^2.$$

Similarly, using (5.5) we have that

$$\tilde{\mu}_2(\lambda) = \frac{2(1-a\lambda-a^2\lambda^2(1-a\lambda))}{\lambda^3(1-a\lambda)^2} = \frac{2(1+a\lambda)}{\lambda^3} = 2/\lambda^3 + 2a/\lambda^2$$

This yields that $\mu(t) = t$ and

$$\sigma^2(t) = t^2 + 2at - t^2 = 2at.$$

6. Moments of some subordinated processes

In this section we point out that we can use the moments of the Hitting time distribution to estimate the moments for any process with known moments that is subordinated against the Hitting time distribution. So if the semigroup orbit $T(t)f$ has mean $\mu(t)$ and variance $\sigma^2(t)$, then the subordinated process

$$S(t,x) = \int_0^\infty T(s)f(x)\, d_s(H(t,s)),$$

has mean

$$\mu_S(t)_i := \int x_i S(t,x)\, dx = \int_0^\infty \mu_i(s) d_s H(t,s) \tag{6.1}$$

and covariance
(6.2)

$$\sigma_S^2(t)_{i,j} := \int x_i x_j S(t,x)\,dx - \mu_S(t)_i \mu_S(t)_j = \int_0^\infty \sigma_{i,j}^2(s) + \mu_i(s)\mu_j(s)\,d_s H(t,s) - \mu_S(t)_i \mu_S(t)_j.$$

6.1. **Pure Diffusion.** In pure diffusion, $\mu(t) = \mu_S(t) = 0$ and $\sigma^2(t) = Dt$ for some diffusion covariance matrix D.

In the case of $\gamma < 1$, by virtue of (5.1), the variance will grow like

$$\sigma_S^2(t) = Dt^\gamma / \Gamma(1+\gamma).$$

In the case of $1 < \gamma < 2$, using (5.4), we obtain that the variance grows like

$$\sigma_S^2(t) \approx D(t + at^{2-\gamma}/\Gamma(3-\gamma) + -...).$$

In practice this may be mistaken for slower than linear growth.

A surprising consequence is that for pure diffusion in the case of $\gamma = 2$, the variance in time, a, has no bearing on the resulting variance:

$$\sigma_S^2(t) = Dt.$$

However, the tailing is affected by the parameter a.

6.2. **Delayed advection and dispersion.** In case $T(t)$ is the classical advection-dispersion semigroup, we have that $\mu(t) = vt$ for some velocity vector v, and $\sigma^2(t) = Dt$ for a dispersion covariance matrix D.

In case $\gamma < 1$, the delay affects the mean plume location. Substituting (5.1) into (6.1) we obtain

$$\mu_S(t) = vt^\gamma / \Gamma(1+\gamma).$$

The effect on the variance is surprising as for $\gamma > 0.5$ the velocity vector gives rise to a super-diffusive process:

$$\sigma_S^2(t)_{i,j} = D_{ij} t^\gamma / \Gamma(1+\gamma) + v_i v_j t^{2\gamma} (2/\Gamma(1+2\gamma) - 1/(\Gamma(1+\gamma))^2).$$

In case $1 < \gamma < 2$, we obtain

$$\mu_S(t) = v(t + at^{2-\gamma}/\Gamma(3-\gamma) + ...).$$

For the covariance we compute

$$\sigma_S^2(t)_{i,j} = D_{ij}(t + at^{2-\gamma}/\Gamma(3-\gamma)) + v_i v_j (at^{3-\gamma}(4/\Gamma(4-\gamma) - 2/\Gamma(3-\gamma)) + c_2 t^{4-2\gamma} + -...)$$

For $\gamma = 2$ the covariance grows linearly,

$$\sigma_{i,j}^2(t) = D_{ij} t + 2 v_i v_j a t + v_i v_j t^2 - v_i v_j t^2 = (D_{ij} + 2 a v_i v_j) t.$$

Note that the highest order terms in the respective covariances are proportional to the square of the velocity, confirming the argument the the high rate of dispersion is due to a smearing out effect rather than an inherent super-diffusion.

References

[1] L. Ahlfors (1978) *Complex Analysis. An introduction to the theory of analytic functions of one complex variable.* 3rd Ed., McGraw-Hill, New York.

[2] W. Arendt, C.J.K. Batty, M.Hieber, and F.Neubrander (2001) *Vector-valued Laplace transforms and Cauchy problems* Birkhaeuser Verlag, Basel-Boston-Berlin.

[3] B. Baeumer, D.A. Benson, and M. M. Meerschaert (2003) *Advection and Dispersion in Time and Space*. Submitted.

[4] B. Baeumer and M. Meerschaert (2001) Stochastic solutions for fractional Cauchy problems. *Fractional Calculus and Applied Analysis* **4**, 481–500.

[5] G. Baxter and M.D. Donsker (1957) On the distribution of the supremum functional for processes with stationary independent increments. *Trans. Amer. Math. Soc.* **85**, 73–87.

[6] P. Becker-Kern, M.M. Meerschaert and H.P. Scheffler (2003) Limit theorem for continuous time random walks with two time scales, *Journal of Applied Probability*, to appear. Available at http:// unr.edu/ homepage/ mcubed/ gammagt1.pdf

[7] M. Caputo (1967) Linear models of dissipation whose Q is almost frequency independent, Part II. *Geophys. J. R. Astr. Soc.* **13**, 529-539.

[8] R. Haggerty, S.A. McKenna, and L.C. Meigs (2000) On the late-time behavior of tracer test breakthrough curves, *Water Resour. Res.* **36**, 3467–3479.

[9] M.M. Meerschaert, D.A. Benson and B. Baeumer (1999) Multidimensional advection and fractional dispersion. *Phys. Rev. E* **59**, 5026–5028.

[10] M.M. Meerschaert, D.A. Benson and B. Baeumer (2001) Operator Lévy motion and multiscaling anomalous diffusion. *Phys. Rev. E* **63**, 1112–1117.

[11] M.M. Meerschaert and H.P. Scheffler (2001) *Limit Distributions for Sums of Independent Random Vectors*. Wiley, New York.

[12] M.M. Meerschaert and H.P. Scheffler (2001) Limit Theorems for Continuous Time Random Walks, submitted. Available at http://unr.edu/ homepage/ mcubed/ LimitCTRW.pdf

[13] M.M. Meerschaert, D.A. Benson, H.P. Scheffler and B. Baeumer (2002) Stochastic solution of space-time fractional diffusion equations. *Phys. Rev. E* **65**, 1103–1106.

[14] R. Metzler and J. Klafter (2000) The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **339**, 1–77.

[15] I. Podlubny (1999) *Fractional differential equations.* Academic Press, San Diego.

[16] R. Remmert (1991) *Theory of Complex functions.* Graduate Texts in Mathematics, **122**. Springer-Verlag, New York.

[17] W. Rudin (1987) *Real and Complex Analysis.* 3rd Ed., McGraw-Hill, New York.

[18] W. Rudin (1991) *Functional Analysis.* 2nd Ed., McGraw-Hill, New York.

[19] V.V. Uchaikin and V.M. Zolotarev (1999) *Chance and stability. Stable distributions and their applications.* VSP, Utrecht.

[20] D.V. Widder (1946) *The Laplace Transform.* Princeton University Press.

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