Subordinated multiparameter groups of linear operators: Properties via the transference principle

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Dedicated to Gunter Lumer;
You were quite an inspiration!

Abstract. In this article we explore properties of subordinated d-parameter groups. We show that they are semi-groups, inheriting the properties of the subordinator via a transference principle. Applications range from infinitely divisible processes on a torus to the definition of inhomogeneous d-dimensional fractional derivative operators.

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1. Introduction

Let S be a strongly continuous uniformly bounded semigroup on $L_1(\mathbb{R}^d)$ that commutes with the translation operator $(T_af)(x) = f(x+a)$ for all $a \in \mathbb{R}^d$ and let $\mathcal{B}(X)$ denote the algebra of bounded linear operators on a general Banach space X. Then, by [11, Theorem 1.4], S is given by

$$[S(t)f](x) = \int_{\mathbb{R}^d} f(x-s)\,\mu_S^t(ds), \;\; f \in L_1(\mathbb{R}^d) \tag{1.1}$$

where $\{\mu_S^t\}_{t\geq 0}$ is a family of bounded complex regular measures on $\mathbb{R}^d$ with

$$||S||_{\mathcal{B}(L_1(\mathbb{R}^d))} = |\mu_S^t|(\mathbb{R}^d) \leq M_S \tag{1.2}$$

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for all $t \geq 0$. The Fourier transform of $S(t)f$ is given by

$$\widehat{S(t)f}(k) = \int_{\mathbb{R}^d} e^{-i\langle k,s\rangle} S(t)f(s)\, ds = e^{t\psi(k)} \hat{f}(k). \tag{1.3}$$

If, in addition, S is also positive ; i.e., $S(t)f \geq 0$ for all $f, t \geq 0$, then S has a Lévy-Khintchine representation (see, for example, [13]), namely ψ is given via[1]

$$\psi(k) = -c^2 - i\langle k, a\rangle - \frac{1}{2}\langle k, Qk\rangle + \int_{x\neq 0} \Big(e^{-i\langle k,x\rangle} - 1 + \frac{i\langle k, x\rangle}{1+|x|_2^2} \Big)\, \phi(dx) \tag{1.4}$$

where $a \in \mathbb{R}$, $Q = \{q_{ij}\}_{i,j=1}^d$ is a symmetric non-negative definite $d \times d$ matrix with real entries, and the *Lévy measure* ϕ is a σ-finite Borel measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{x\neq 0} \min\{1, |x|_2^2\}\, \phi(dx) < \infty. \tag{1.5}$$

Let G be a d-parameter C_0-group of operators on a Banach space X generated by $\{(A_i, \mathcal{D}(A_i)) : i = 1 \ldots d\}$. In this article we investigate the properties of semigroups obtained by subordinating G by S; i.e. we investigate

$$G_S(t)x = \int_{\mathbb{R}^d} G(s)x\, \mu_S^t(ds),\ \ x \in X, t \geq 0. \tag{1.6}$$

We develop a general theory of these subordinated d-parameter groups, including a powerful transference principle Theorem 2.8 that can be used to show how the subordinated group inherits many useful properties of the subordinator S in $L_1(\mathbb{R}^d)$.

In particular, we first show that G_S is indeed a bounded semigroup and give a core for its infinitesimal generator A_S. Let $(\mathcal{M}_\psi, \mathcal{D}(\mathcal{M}_\psi))$ denote the generator of S on $L_1(\mathbb{R}^d)$ with Fourier transform (1.3). We then prove a transference principle; i.e., we establish that

$$\|g(A_S)\|_{\mathcal{B}(X)} \leq C \|g(\mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))}$$

for all allowable functions g in the Hille-Phillips functional calculus. The constant C does *not* depend on g. We thereby show, for example, how regularity of S translates into regularity of G_S.

Of special interest to applications is the case where S is positive and $\|S\| = 1$. Note that if $\psi(0) = 0$, then, using (1.1) and (1.2), it is easy to see that the two properties are equivalent. Then subordination has a stochastic interpretation as randomising time (velocity) in each component against an infinitely divisible distribution and ψ is given by the Lévy-Khintchine formula above with $c = 0$. If $\{(A_i, \mathcal{D}(A_i)),\ i = 1, 2, \ldots, d\}$ denotes the set of generators of the multi-parameter

[1]Throughout the paper for $x \in \mathbb{R}^d$ we denote $|x|_2 := \Big(\sum_{i=1}^d |x_i|^2\Big)^{1/2}$ in order to distinguish it to the norm in an arbitrary Banach space.

group G, we give in Theorem 2.12 the proof of an explicit generator formula for all $x \in \bigcap_{i,j=1}^{d} \mathcal{D}(A_i A_j)$; i.e.,

$$A_S x = \sum_{i=1}^{d} a_i A_i x + \frac{1}{2} \sum_{i,j=1}^{d} q_{ij} A_i A_j x + \int_{s \neq 0} \Big(G(s)x - x - \sum_{i=1}^{d} \frac{s_i A_i x}{1 + |s|_2^2} \Big) \phi(ds),$$

extending the result of Phillips [20]. In case that S is unilateral; i.e., for all $t \geq 0$, $\mu_S^t(\Omega \cap \mathbb{R}_i^{d-}) = 0$ for all measurable $\Omega \subset \mathbb{R}^d$ and all $\mathbb{R}_i^{d-} := \{s \in \mathbb{R}^d : s_i \leq 0\}$, our theory readily applies to subordinating semi-groups and we generalise $d = 1$ results by Phillips [20], Carasso and Kato [6], Berg, Boyadzhiev and DeLaubenfels [4] and Baeumer and Kovács [2].

2. The transference principle and generator formulas

Since the treatment of multi-parameter semigroups and groups are not standard we first summarise some of their basic properties in the following proposition (see, for example, [5, Propositions 1.1.8 and 1.1.9]).

Proposition 2.1. *If T is a d-parameter C_0-semigroup on X, then T is the product of d one-parameter C_0-semigroups T_i with generators $(A_i, \mathcal{D}(A_i))$; i.e., for $t = (t_1, \ldots, t_d)$ we have $T(t) = \prod_{i=1}^{d} T_i(t_i)$ and the operators $T_i(t_i)$ commute with each other, $0 \leq t_i < \infty$ and $i = 1, \ldots, d$. Moreover,*

- (i) *If $x \in \mathcal{D}(A_i)$ then $T(t)x \in D(A_i)$ and $A_i T(t)x = T(t)A_i x$, $t \in \mathbb{R}_+^d$.*
- (ii) *The set $\bigcap_{i=1}^{d} \mathcal{D}(A_i)$ is a dense subspace of X and furthermore a Banach space with norm $||x||_{\bigcap_{i=1}^{d} \mathcal{D}(A_i)} := ||x|| + \sum_{i=1}^{d} ||A_i x||$.*
- (iii) *If $x \in \mathcal{D}(A_i)$ and $x \in \mathcal{D}(A_i A_j)$, then $x \in \mathcal{D}(A_j A_i)$ and $A_j A_i x = A_i A_j x$.*

For a d-parameter C_0-semigroup T on X the set $\{(A_i, \mathcal{D}(A_i)),\ i = 1, 2, \ldots, d\}$ is called the *set of generators.*

Recall that X is a Banach space and G a bounded d-parameter C_0-group on X with set of generators $\{(A_i, \mathcal{D}(A_i)),\ i = 1, \ldots, d\}$. We define the *subordinated group* G_S via equation

$$G_S(t)x = \int_{\mathbb{R}^d} G(s)x\, \mu_S^t(ds),\ x \in X, t \geq 0.$$

where $\mu_S^t(ds)$ is given by (1.1). We will show first that G_S is a bounded C_0-semigroup. The proof (and basically all of the proofs for the theorems that follow) is based on the following commonly used construction.

Definition 2.2. Let G be a bounded d-parameter C_0 -group on X, $f \in L_1(\mathbb{R}^d)$, and $x \in X$. We say that an element $x_f \in X$ is *G-mollified* or short *mollified* if it is obtained by mollifying $\mathbb{R}^d \ni t \mapsto G(t)x$ with f ; i.e., if

$$x_f = \int_{\mathbb{R}^d} f(r)G(r)x\, dr. \tag{2.1}$$

The following Lemma shows that the set of mollified elements is sufficiently rich for our purposes.

Lemma 2.3. *Consider the set of mollified elements* $\mathcal{B} := \{x_f, x \in Y, f \in \mathcal{C} \subset L_1(\mathbb{R}^d)\}$*. If* $\mathcal{C}$ *is dense in* $L_1(\mathbb{R}^d)$ *and* Y *is dense in* X*, then* $\mathcal{B}$ *is dense in* X*.*

Proof. Assume that $\mathcal{B}$ is not dense in X. Then there is $0 \neq x^* \in X^*$ such that $\langle x_f, x^* \rangle = 0$ for all $x_f \in \mathcal{B}$; i.e,

$$\langle \int_{\mathbb{R}^d} G(s)xf(s)\,ds, x^* \rangle = \int_{\mathbb{R}^d} \langle G(s)x, x^* \rangle f(s)\,ds = 0,\ \forall f \in \mathcal{C},\ x \in Y.$$

Now $\mathbb{R}^d \ni s \mapsto \langle G(s)x, x^* \rangle$ is continuous and bounded and hence belongs to $(L_1(\mathbb{R}^d))^* = L_\infty(\mathbb{R}^d)$. This implies that if $\mathcal{C}$ is dense in $L_1(\mathbb{R}^d)$, $\langle G(s)x, x^* \rangle \equiv 0$ for all $x \in Y$ and $s \in \mathbb{R}^d$. But $G(0) = I$ and therefore $\langle x, x^* \rangle = 0$ for all $x \in Y$ which implies that Y cannot be dense in X as $x^* \neq 0$. Hence $\mathcal{C}$ and Y being dense implies that $\mathcal{B}$ has to be dense. □

Proposition 2.4. G_S *given by (*1.6*) is a bounded* C_0*-semigroup on* X*.*

Proof. The operator family G_S is well defined since $\mathbb{R}^d \ni s \mapsto G(s)x$ is continuous and μ_S^t is a bounded measure. If G is bounded by M_G, then G_S is bounded by $M_S M_G$, where M_S is the constant from (1.2). Let

$$\mathcal{A} := \{x_f,\ x \in X, f \in L_1(\mathbb{R}^d)\} \tag{2.2}$$

be the set of mollified elements. By Fubini's theorem, for $x_f \in \mathcal{A}$,

$$\begin{aligned} G_S(t)x_f &= \int_{\mathbb{R}^d} G(s) \int_{\mathbb{R}^d} f(r)G(r)x\,dr\,\mu_S^t(ds) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(r)G(s+r)x\,dr\,\mu_S^t(ds) \\ &= \int_{\mathbb{R}^d} G(v)x \int_{\mathbb{R}^d} f(v-s)\,\mu_S^t(ds)\,dv \\ &= \int_{\mathbb{R}^d} [S(t)f](v)G(v)x\,dv = x_{S(t)f}. \end{aligned} \tag{2.3}$$

This shows that for $x_f \in \mathcal{A}$,

$$G_S(0)x_f = x_{S(0)f} = x_f,$$

and

$$G_S(t+s)x_f = x_{S(t+s)f} = x_{S(t)S(s)f} = G_S(t)x_{S(s)f} = G_S(t)G_S(s)x_f.$$

The map $L_1(\mathbb{R}^d) \ni f \mapsto x_f \in X$ is clearly linear, and since

$$\|x_f\| \leq M_G \|f\|_{L_1(\mathbb{R}^d)} \|x\|, \tag{2.4}$$

it is also continuous. Therefore,

$$\begin{aligned} \|G_S(t)x_f - x_f\| =& \|x_{S(t)f} - x_f\| = \|x_{S(t)f-f}\| \\ \leq& M_G \|S(t)f - f\|_{L_1(\mathbb{R}^d)} \|x\|, \end{aligned} \tag{2.5}$$

and since S is a C_0-semigroup on $L_1(\mathbb{R}^d)$, we obtain that $t \to G_S(t)x_f$ is continuous at $t = 0+$. Finally, by Lemma 2.3, $\mathcal{A}$ is dense in X and since $\|G_S(t)\| \le M_S M_G$, $t \ge 0$, the above holds for all $x \in X$ and the proof is complete. □

Next we identify a reasonably large subset, i.e., a core of the domain of the generator A_S of G_S.

Theorem 2.5. *The set*

$$\mathcal{C} := \{x_f,\ x \in X,\ f \in \mathcal{D}(\mathcal{M}_\psi)\} \subset \mathcal{D}(A_S)$$

is a core for A_S and

$$A_S x_f = x_{\mathcal{M}_\psi f},\ x_f \in \mathcal{C},\ \textit{with } \|A_S x_f\| \le M_G \|\mathcal{M}_\psi f\|_{L_1(\mathbb{R}^d)} \|x\|.$$

Proof. Let $x_f \in \mathcal{C}$. Then $f \in \mathcal{D}(\mathcal{M}_\psi)$ and

$$\begin{aligned}\left\| \frac{G_S(h)x_f - x_f}{h} - x_{\mathcal{M}_\psi f} \right\| &= \left\| x_{\left(\frac{S(h)f - f}{h} - \mathcal{M}_\psi f\right)} \right\| \\ &\le M_G \left\| \frac{S(h)f - f}{h} - \mathcal{M}_\psi f \right\|_{L_1(\mathbb{R}^d)} \|x\| \to 0\end{aligned}$$

as $h \searrow 0$. This proves that $x_f \in \mathcal{D}(A_S)$ and $A_S x_f = x_{\mathcal{M}_\psi f}$. By (2.5) we see that $\|A_S x_f\| \le M_G \|\mathcal{M}_\psi f\|_{L_1(\mathbb{R}^d)} \|x\|$. If $f \in \mathcal{D}(\mathcal{M}_\psi)$, then $S(t)f \in \mathcal{D}(\mathcal{M}_\psi)$. Therefore, in view of (2.3), G_S leaves $\mathcal{C}$ invariant. Also, $\mathcal{D}(\mathcal{M}_\psi)$ is dense in $L_1(\mathbb{R}^d)$ and thus $\mathcal{C}$ is dense in X by Lemma 2.3. Therefore, $\mathcal{C}$ is a core for A_S by [8, Chapter II, Proposition 1.7]. □

As a consequence we can transfer the action of A_i on mollified elements of X to actions on their mollifiers.

Corollary 2.6. *Let $f \in W^{2,1}(\mathbb{R}^d)$, $x \in X$. Then for all $i = 1 \dots d$, $x_f \in \mathcal{D}(A_i)$ and*

$$A_i x_f = -\int_{\mathbb{R}^d} \frac{\partial f(s)}{\partial s_i} G(s)x\, ds = x_{-\frac{\partial f}{\partial s_i}}.$$

Furthermore, for all $i, j = 1, \dots, d$, $x_f \in \mathcal{D}(A_i A_j) \cap \mathcal{D}(A_j A_i)$ and

$$A_i A_j x_f = A_j A_i x_f = \int_{\mathbb{R}^d} \frac{\partial^2 f(s)}{\partial s_i \partial s_j} G(s)x\, ds = x_{\frac{\partial^2 f}{\partial s_i \partial s_j}}.$$

Proof. It is well known that for the ith-coordinate wise right-translation semigroup

$$[T_{r,i}(t)f](s_1, \dots, s_i, \dots, s_n) = f(s_1, \dots, s_i - t, \dots, s_n)$$

acting on $L_1(\mathbb{R}^d)$ the generator is given by $\mathcal{M}_\psi f = -\frac{\partial f}{\partial x_i}$ with domain $\mathcal{D}(\mathcal{M}_\psi) = \{f \in L_1(\mathbb{R}^d) :\ x_i \mapsto f(x_1, \dots, x_i, \dots, x_n) \text{ abs. cont. with } -\frac{\partial f}{\partial x_i} \in L_1(\mathbb{R}^d)\}$. On the other hand, $T_{r,i}$ can be represented by equation (1.1), where $\mu_S^t = \delta_0 \times \cdots \times \delta_t \times \dots \delta_0$ where δ_t is the Dirac measure on $\mathbb{R}$ concentrated at t. Hence $S = T_{r,i}$, $G_S = G$, $A_S = A_i$ and the first statement follows directly from Theorem 2.5. The proof is easily completed by repeating the above argument. □

Now we are in a position to prove our main theorem, the transference of the Hille-Phillips functional calculus, which has remarkable consequences. The transference principle is a powerful tool which has been used in many fields of analysis. For a general reference, see [7]. We recall a form of the Hille-Phillips functional calculus briefly. Let $(C, \mathcal{D}(C))$ generate a C_0-semigroup T_C bounded by $M \geq 1$. For a function $g : \mathbb{C} \hookrightarrow \mathbb{C}$ with representation

$$g(z) := \int_0^\infty e^{zt}\, d\alpha(t) \ (\operatorname{Re} z \leq 0) \tag{2.6}$$

where $\alpha : [0, \infty) \to \mathbb{C}$ is a normalised[2] function of bounded total variation, define

$$g(C)x := \int_0^\infty T_C(t)x\, d\alpha(t),\ \ x \in X$$

where the integral can be understood either in the Riemann-Stieltjes or in the Lebesgue-Stieltjes (Bochner) sense. In the latter case α is then replaced by the complex Borel measure induced by α. It turns out that functions with representation (2.6) form an algebra and the map $\Psi : g \mapsto g(C) \in \mathcal{B}(X)$ is an algebra homomorphism (see, for example, [10, Chapter XV] and [15]), where $\mathcal{B}(X)$ is the algebra of bounded linear operators on X. In particular, for $g_\lambda(z) = 1/(\lambda - z)$ and each λ with $\operatorname{Re}\lambda > 0$, the resolvent is an element of the algebra; i.e., for $x \in X$ and $\operatorname{Re}\lambda > 0$,

$$g_\lambda(C)x = R(\lambda, C)x = \int_0^\infty T_C(t)xe^{-\lambda t}\, dt.$$

Definition 2.7. A collection of functions $\{g_n\}_{n\in\mathbb{N}} \subset L_1(\mathbb{R}^d)$ is called an *approximate identity* if $g_n \geq 0$, $\|g_n\|_{L_1(\mathbb{R}^d)} = 1$ and $\lim_{n\to\infty} \int_{|s|_2 \geq \delta} g_n(s)\, ds = 0$ for any fixed $\delta > 0$.

It is straightforward to show that if $x \in X$ and $\{f_n\}_{n\in\mathbb{N}}$ is an approximate identity, then $x_{f_n} \to x$ as $n \to \infty$.

Theorem 2.8 (Transference Principle). *Let A_S be the generator of the semigroup G_S and $\mathcal{M}_\psi$ be the generator of the semigroup S. If g has representation (2.6), then*

$$\|g(A_S)\|_{\mathcal{B}(X)} \leq M_G \|g(\mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))} \tag{2.7}$$

where M_G is independent of g.

[2]One may, for example, normalise α by setting $\alpha(0) = 0$ and $\alpha(u) = \frac{\alpha(u+)+\alpha(u-)}{2}$ for all $u > 0$, see, for example [15],

Proof. Let $x_f \in \mathcal{A}$ where $\mathcal{A}$ is defined in (2.2) and g given by (2.6). Then, since both G_S and S are bounded C_0-semigroups,

$$\begin{aligned} g(A_S)x_f &= \int_0^\infty G_S(t)x_f\, d\alpha(t) = \int_0^\infty x_{S(t)f}\, d\alpha(t) \\ &= \int_0^\infty \int_{\mathbb{R}^d} [S(t)f](r)G(r)x\, dr\, d\alpha(t) \\ &= \int_{\mathbb{R}^d} \int_0^\infty [S(t)f](r)\, d\alpha(t)G(r)x\, dr = x_{g(\mathcal{M}_\psi)f}. \end{aligned} \tag{2.8}$$

where the interchange of the integrals is justified by Fubini's theorem. Therefore, by (2.4),

$$\begin{aligned} \|g(A_S)x_f\| &\le M_G \|g(\mathcal{M}_\psi)f\|_{L_1(\mathbb{R}^d)}\|x\| \\ &\le M_G\|g(\mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))}\|f\|_{L_1(\mathbb{R}^d)}\|x\|. \end{aligned}$$

Finally, take $\{f_n\}_{n\in\mathbb{N}}$ to be an approximate identity. Then $x_{f_n} \to x$ and since $g(A_S) \in \mathcal{B}(X)$ we also have that $g(A_S)x_{f_n} \to g(A_S)x$ for all $x \in X$. Thus,

$$\|g(A_S)x\| \le M_G\|g(\mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))}\|x\|, \ x \in X.$$

□

We immediately obtain an important corollary which shows an example of transference of regularity under subordination in the group case. Recall that a C_0-semigroup T_C generated by $(C, \mathcal{D}(C))$ is called a *bounded analytic semigroup of angle* $\theta \in (0, \frac{\pi}{2}]$ if T_C has a bounded analytic extension to a sectorial region $\{z \in \mathbb{C} : |\arg z| < \theta'\}$ for all $\theta' \in (0, \theta)$. This is equivalent to $(C, \mathcal{D}(C))$ being *a sectorial operator of angle* θ; that is, the resolvent set of C, $\rho(C)$, contains the sectorial region

$$\Sigma_\theta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \theta\} \setminus \{0\} \subset \rho(C)$$

and $\|R(\lambda, C)\|_{\mathcal{B}(X)} \le \frac{M_\varepsilon}{|\lambda|}$ for all $\lambda \in \Sigma_{\theta-\varepsilon}$ and $\varepsilon \in (0, \delta)$ for some $M_\varepsilon \ge 1$ (see, for example [1, Theorem 3.7.11]).

Corollary 2.9. *If S is a bounded analytic semigroup of angle θ on $L_1(\mathbb{R}^d)$, then G_S is a bounded analytic semigroup of angle θ on X.*

Proof. Since both G_S and S are bounded semigroups it follows that $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ is contained in both, the resolvent set $\rho(A_S)$ and $\rho(\mathcal{M}_\psi)$. If $g(z) := (\lambda - z)^{-1}$ $(\operatorname{Re}\lambda > 0)$, then for the resolvent operators of A_S and $\mathcal{M}_\psi$ we obtain, by Theorem 2.8,

$$\|R(\lambda, A_S)\|_{\mathcal{B}(X)} \le M_G\|R(\lambda, \mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))}, \ \operatorname{Re}\lambda > 0. \tag{2.9}$$

Therefore,

$$\sup_{\operatorname{Re}\lambda>0} \|\lambda R(\lambda, A_S)\|_{\mathcal{B}(X)} \le M_G \sup_{\operatorname{Re}\lambda>0} \|\lambda R(\lambda, \mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}^d))} < \infty$$

where the finite supremum on the right hand side follows from the assumption that S is an analytic semigroup on $L_1(\mathbb{R}^d)$ (see, for example [1, Corollary 3.7.12]). Even more, the finiteness of above supremum is a necessary and sufficient condition for the analyticity of a semigroup [1, Corollary 3.7.12]. Thus G_S is indeed a bounded analytic semigroup . Assume that S is a bounded analytic semigroup of angle θ, or, equivalently, that $\mathcal{M}_\psi$ is a sectorial operator of angle θ. Let $\lambda \in \Sigma_{\theta-\varepsilon}$. Then there is $\lambda_0 \in i\mathbb{R}$ and $r > 0$ such that $\lambda \in D(\lambda_0, r) \subset \rho(\mathcal{M}_\psi)$, where $D(\lambda_0, r)$ is the open disc with centre λ_0 and radius r. It follows from (2.9) that $\lambda_0 \in \rho(A_S)$. Therefore, by Theorem 2.8, the algebra homomorphism property of the Hille-Phillips functional calculus and the continuity of the resolvent,

$$\left\|R(\lambda_0, A_S)^{k+1}\right\|_{\mathcal{B}(X)} \leq M_G \left\|R(\lambda_0, \mathcal{M}_\psi)^{k+1}\right\|_{\mathcal{B}(L_1(\mathbb{R}^d))}.$$

Hence absolute convergence of $\sum(\lambda - \lambda_0)^k R(\lambda_0, \mathcal{M}_\psi)^{k+1}$ implies absolute convergence of $\sum(\lambda-\lambda_0)^k R(\lambda_0, A_S)^{k+1}$. Therefore, using the Taylor series representation of the resolvent, $\lambda \in \rho(A_S)$ and $\|R(\lambda, A_S)\|_{\mathcal{B}(X)} \leq \frac{M_\varepsilon M_G}{|\lambda|}$. ☐

2.1. Positive Subordinators

Of particular interest is the case where the subordinator S is positive as it applies to stochastic models and the subordinator takes on the purpose of randomising the time variable (or velocity) in each component. In this case we prove an explicit generator formula using rather elementary tools. We need two preparatory results.

Proposition 2.10. *Let T be a d-parameter C_0-semigroup on a Banach space X with set of generators $\{(A_i, \mathcal{D}(A_i))\ i = 1, \dots, d\}$. A subspace of $\bigcap_{i=1}^d \mathcal{D}(A_i)$ which is $\|\cdot\|$-dense in X and invariant under T is $\|\cdot\|_{\bigcap_{i=1}^d \mathcal{D}(A_i)}$-dense in $\bigcap_{i=1}^d \mathcal{D}(A_i)$.*

Proof. The proof is a straightforward generalisation of the proof of the well known one-parameter result (see, for example, [8, Chapter II, Proposition 1.7]) in view of Proposition 2.1. ☐

Consider $\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)$ with norm

$$||x||_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)} := ||x|| + \sum_{i=1}^d ||A_ix|| + \sum_{i,j=1}^d ||A_iA_jx||.$$

Note that, by Proposition 2.1 (iii), similarly to Sobolev spaces, one could use an equivalent norm by summing up the last term for $1 \leq j \leq i \leq d$, only.

Corollary 2.11. *Let T be a d-parameter C_0-semigroup on a Banach space X with set of generators $\{(A_i, \mathcal{D}(A_i))\ i = 1, \dots, d\}$. A subspace $\mathcal{D}$ of $X_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)}$ which is $\|\cdot\|$-dense in X and invariant under T is $\|\cdot\|_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)}$-dense in $X_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)}$.*

Proof. Clearly, T is a d-parameter C_0-semigroup on the Banach space

$$\left(\bigcap_{i=1}^d \mathcal{D}(A_i), \|\cdot\|_{\bigcap_{i=1}^d \mathcal{D}(A_i)}\right)$$

in view of Proposition 2.1, with set of generators $(A_i, \bigcap_{j=1}^d \mathcal{D}(A_iA_j))$. Therefore, by Proposition 2.10, $\mathcal{D}$ is dense in $\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)$ with respect to the norm

$$|||x||| = \|x\|_{\bigcap_{i=1}^d \mathcal{D}(A_i)} + \sum_{j=1}^d \|A_jx\|_{\bigcap_{i=1}^d \mathcal{D}(A_i)} \geq \|x\|_{\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)},$$

which finishes the proof. □

Next we prove the generator formula.

Theorem 2.12. *Let S be positive; i.e, the log-characteristic function is given by the Lévy-Khintchine formula* (1.4). *Then* $\bigcap_{i,j=1}^d \mathcal{D}(A_iA_j) \subset \mathcal{D}(A_S)$ *and*

$$A_Sx = -c^2x+\sum_{i=1}^d a_iA_ix+\frac{1}{2}\sum_{i,j=1}^d q_{ij}A_iA_jx+\int_{s\neq 0}\Big(G(s)x-x-\sum_{i=1}^d \frac{s_iA_ix}{1+|s|_2^2}\Big)\,\phi(ds), \tag{2.10}$$

for all $x \in \bigcap_{i,j=1}^d \mathcal{D}(A_iA_j)$.

Proof. In [3, Theorem 2.2] it is shown that $W^{2,1}(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{M}_\psi)$ and

$$\begin{aligned}(\mathcal{M}_\psi f)(s) = &-c^2 f(s) - a\cdot\nabla f(s) + \tfrac{1}{2}\,\nabla\cdot Q\nabla f(s)\\ &+\int_{y\neq 0}\Big(f(s-y)-f(s)+\frac{\nabla f(s)\cdot y}{1+|y|_2^2}\Big)\,\phi(dy),\ \ f\in W^{2,1}(\mathbb{R}^d).\end{aligned}$$

Let $\mathcal{D} := \{x_f : x\in X,\ f\in W^{2,1}(\mathbb{R}^d)\cap C^2(\mathbb{R}^d)\}$. By Theorem 2.5, $\mathcal{D}\subset\mathcal{D}(A_S)$. For $x_f\in\mathcal{D}$ we have, using Corollary 2.6,

$$\begin{aligned}A_Sx_f &= x_{\mathcal{M}_\psi f}\\ &= \int_{\mathbb{R}^d}\big[-c^2 f(s) - a\cdot\nabla f(s) + \tfrac{1}{2}\,\nabla\cdot Q\nabla f(s)\\ &+\int_{y\neq 0}\Big(f(s-y)-f(s)+\frac{\nabla f(s)\cdot y}{1+|y|_2^2}\Big)\ \phi(dy)\big]\,G(s)x\,ds\\ &= -c^2x_f+\sum_{i=1}^d a_iA_ix_f+\frac{1}{2}\sum_{i,j=1}^d q_{ij}A_iA_jx_f\\ &+\int_{\mathbb{R}^d}\Big(\int_{v\neq 0}\Big(f(s-v)-f(s)+\frac{\nabla f(s)\cdot v}{1+|v|_2^2}\Big)\,\phi(dv)\Big)\,G(s)x\,ds\\ &= -c^2x_f+\sum_{i=1}^d a_iA_ix_f+\frac{1}{2}\sum_{i,j=1}^d q_{ij}A_iA_jx_f\\ &+\int_{v\neq 0}\Big(G(v)x_f-x_f-\sum_{i=1}^d\frac{v_iA_ix_f}{1+|v|_2^2}\Big)\ \phi(dv),\end{aligned} \tag{2.11}$$

where the interchange of the integrals is justified because $\|G(s)x_f\| \le M_G\|x_f\|$ and because $f \in W^{2,1}(\mathbb{R}^d) \cap C^2(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{M}_\psi)$ and thus

$$s \mapsto \int_{v\neq 0} \left(f(s-v) - f(s) + \frac{\nabla f(s)\cdot v}{1+|v|_2^2} \right) \phi(dv) \in L_1(\mathbb{R}^d).$$

Taylor's formula for $f \in W^{2,1}(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$ yields

$$f(s-v) = f(s) - v\cdot\nabla f(s) + \int_0^1 (1-t)\langle v, M_{s-tv}v\rangle\, dt$$

where M_r is the Hessian matrix of f at $r \in \mathbb{R}^d$. Thus for $x_f \in \mathcal{D}$, by Corollary 2.6 and Fubini's theorem,

$$\begin{aligned} G(v)x_f &= x_{f(\cdot - v)} = x_{f - v\cdot\nabla f + \int_0^1 (1-t)\langle v, M_{\cdot - tv} v\rangle\, dt} \\ &= x_f + \sum_{i=1}^d v_i A_i x_f + \int_0^1 (1-t) G(tv) \sum_{i,j=1}^d v_i v_j A_i A_j x_f\, dt \end{aligned}$$

Hence, for $|v|_2 \le 1$,

$$\begin{aligned} &\left\| G(v)x_f - x_f - \sum_{i=1}^d \frac{v_i A_i x_f}{1+|v|_2^2} \right\| \\ &\le \|G(v)x_f - x_f - \sum_{i=1}^d v_i A_i x_f\| + \left\| \sum_{i=1}^d v_i A_i x_f - \sum_{i=1}^d \frac{v_i A_i x_f}{1+|v|_2^2} \right\| \\ &\le \frac{1}{2} M_G \sum_{i,j=1}^d |v_i v_j|\, \|A_i A_j x_f\| + \frac{|v|_2^2}{1+|v|_2^2} \sum_{i=1}^d |v_i|\, \|A_i x_f\| \\ &\le C_d^2 \frac{|v|_2^2}{2} M_G \sum_{i,j=1}^d \|A_i A_j x_f\| + C_d \frac{|v|_2^3}{1+|v|_2^2} \sum_{i=1}^d \|A_i x_f\| \\ &\le C_d^2 \frac{|v|_2^2}{1+|v|_2^2} M_G \sum_{i,j=1}^d \|A_i A_j x_f\| + C_d \frac{|v|_2^2}{1+|v|_2^2} \sum_{i=1}^d \|A_i x_f\| \\ &\le K_d M_G ||x_f||_{\bigcap_{i,j=1}^d \mathcal{D}(A_i A_j)} \frac{|v|_2^2}{1+|v|_2^2} \end{aligned}$$

If $|v|_2 \geq 1$, then

$$\left\|G(v)x_f - x_f - \sum_{i=1}^{d} \frac{v_i A_i x_f}{1+|v|_2^2}\right\| \leq (M_G+1)\|x_f\| + C_d \frac{|v|_2}{1+|v|_2^2} \sum_{i=1}^{d} \|A_i x_f\|$$
$$\leq (M_G+1)\|x_f\| + C_d \frac{|v|_2^2}{1+|v|_2^2} \sum_{i=1}^{d} \|A_i x_f\|$$
$$\leq (M_G+1)\frac{2|v|_2^2}{1+|v|_2^2}\|x\| + C_d \frac{|v|_2^2}{1+|v|_2^2} \sum_{i=1}^{d} \|A_i x_f\|$$
$$\leq L_d(M_G+1)||x_f||_{\bigcap_{i,j=1}^{d}\mathcal{D}(A_iA_j)} \frac{|v|_2^2}{1+|v|_2^2}.$$

Therefore, for all $v \in \mathbb{R}^d$ and $x_f \in \mathcal{D}$,

$$\left\|G(v)x_f - x_f - \sum_{i=1}^{d} \frac{v_i A_i x_f}{1+|v|_2^2}\right\| \leq (C_d+K_d)(M_G+1)||x_f||_{\bigcap_{i,j=1}^{d}\mathcal{D}(A_iA_j)} \frac{|v|_2^2}{1+|v|_2^2}, \tag{2.12}$$

which implies that

$$\int_{v\neq 0} \left\|G(v)x_f - x_f - \sum_{i=1}^{d} \frac{v_i A_i x_f}{1+|v|_2^2}\right\| \phi(dv) \leq (C_d+K_d)(M_G+1) \int_{v\neq 0} \frac{|v|_2^2}{1+|v|_2^2}\phi(dv)||x_f||_{\bigcap_{i,j=1}^{d}\mathcal{D}(A_iA_j)}. \tag{2.13}$$

The set $\mathcal{C}$ is dense in $L_1(\mathbb{R})$ and therefore by Lemma 2.3, $\mathcal{D}$ is dense in X. It is easy to see that G leaves $\mathcal{D}$ invariant and by Corollary 2.6, $\mathcal{D} \subset \bigcap_{i,j=1}^{d}\mathcal{D}(A_iA_j)$. Thus by Corollary 2.11, $\mathcal{D}$ is $\|\cdot\|_{\bigcap_{i,j=1}^{d}\mathcal{D}(A_iA_j)}$-dense in $\bigcap_{i,j=1}^{d}\mathcal{D}(A_iA_j)$. Therefore, if $x \in \bigcap_{i,j=1}^{d}\mathcal{D}(A_iA_j)$, there is a sequence $\{(x_n)_{f_n}\} \subset \mathcal{D}$ such that $(x_n)_{f_n} \to x$, $A_i(x_n)_{f_n} \to A_i x$, $i = 1,\ldots,d$, and $A_iA_j(x_n)_{f_n} \to A_iA_j x$, $i,j = 1,\ldots,d$, as $n \to \infty$. In particular, $\sup_n \|(x_n)_{f_n}\|_{\bigcap_{i,j=1}^{d}\mathcal{D}(A_iA_j)} < \infty$. Therefore, using (2.13) and (2.12), the right hand side of (2.11) converges to

$$-c^2x + \sum_{i=1}^{d} v_i A_i x + \frac{1}{2}\sum_{i,j=1}^{d} q_{ij}A_iA_j x + \int_{s\neq 0}\Big(G(v)x - x - \sum_{i=1}^{d}\frac{v_iA_ix}{1+|v|_2^2}\Big)\,\phi(dv).$$

Since $x_{f_n} \to x$ and A_S is closed, $x \in \mathcal{D}(A_S)$ and

$$A_S x = -\,c^2x + \sum_{i=1}^{d} v_i A_i x + \frac{1}{2}\sum_{i,j=1}^{d} q_{ij}A_iA_j x + \int_{s\neq 0}\Big(G(v)x - x - \sum_{i=1}^{d}\frac{v_iA_ix}{1+|v|_2^2}\Big)\,\phi(dv),\ \ x \in \bigcap_{i,j=1}^{d}\mathcal{D}(A_iA_j).$$

□

2.2. The semigroup case

In case that S is unilateral; i.e. for all $t \geq 0$, $\mu_S^t(\Omega \cap \mathbb{R}_i^{d-}) = 0$ for all measurable $\Omega \subset \mathbb{R}^d$ and all $\mathbb{R}_i^{d-} := \{s \in \mathbb{R}^d : s_i \leq 0\}$, the above theory readily extends to the point where we can allow G to be a bounded d-parameter semigroup T; i.e., we define

$$T_S(t)x = \int_{\mathbb{R}_+^d} T(s)x\, \mu_S^t(ds),\ x \in X, \tag{2.14}$$

with generator $(A_S, \mathcal{D}(A_S))$. We denote, as above, the set of generators of T by $\{(A_i, \mathcal{D}(A_i)),\ i = 1, 2, \ldots, d\}$ and the generator of S, which is now considered as a C_0-semigroup on $L_1(\mathbb{R}_+^d)$, by $(\mathcal{M}_\psi, \mathcal{D}(\mathcal{M}_\psi))$. With the obvious modifications all of the above theorems hold, in particular, we would like to highlight the Transference Principle for semigroups.

Theorem 2.13 (Transference Principle for semi-groups). *If g has the form (2.6), then*

$$\|g(A_S)\|_{\mathcal{B}(X)} \leq M_T \|g(\mathcal{M}_\psi)\|_{\mathcal{B}(L_1(\mathbb{R}_+^d))} \tag{2.15}$$

where M_T is independent of g.

□

The following corollary is more general then the one-dimensional result in [6], as it also shows the transference of the angle of analyticity and is more general then the corresponding one dimensional statement in [4] as there is no restriction on the measures. For the one dimensional case, see also [2].

Corollary 2.14. *If S is a bounded analytic semigroup on $L_1(\mathbb{R}^d)$ of angle θ on $L_1(\mathbb{R}_+^d)$, then T_S is a bounded analytic semigroup of angle θ on X.*

The other special case we would like to highlight, is the case when S is positive and unilateral. Then the Lévy-Khintchine representation simplifies to

$$\psi(k) = -c^2 - i\langle k, a\rangle + \int_{x \in \mathbb{R}_+^d \setminus \{0\}} \left(e^{-i\langle k, x\rangle} - 1\right) \phi(dx) \tag{2.16}$$

with $a \in \mathbb{R}_+^d$ and ϕ is a σ-finite Borel measure on $\mathbb{R}_+^d \setminus \{0\}$ such that

$$\int \min\{1, |x|_2\} \phi\{dx\} < \infty.$$

This result is originally due to Paul Lévy. The proof for $d = 1$ is outlined in Feller [9, XVII.4(c) p. 571] but the proof extends immediately to the multivariate case (see also [20]). The proof of the generator formula below is analogous to the proof of Theorem 2.12 and therefore we only give an outline.

Theorem 2.15. *Let S be positive and unilateral with log characteristic function given by the Lévy-Khintchine formula (2.16). Then $\bigcap_{i=1}^d \mathcal{D}(A_i) \subset \mathcal{D}(A_S)$ and*

$$A_S x = -c^2 x + \sum_{i=1}^d a_i A_i x + \int_{\mathbb{R}_+^d} \Big(T(s)x - x\Big)\,\phi(ds),\ x \in \bigcap_{i=1}^d \mathcal{D}(A_i). \qquad (2.17)$$

Proof. The proof, the same way as in the group case, uses the result on $L_1(\mathbb{R}_+^d)$, namely, $W_0^{1,1}(\mathbb{R}_+^d) \subset \mathcal{D}(\mathcal{M}_\psi)$ and

$$\mathcal{M}_\psi f(s) = -c^2 f(s) - a \cdot \nabla f(s) + \int_{\mathbb{R}_+^d} (f(s-y) - f(s))\,\phi(dy),\ f \in W_0^{1,1}(\mathbb{R}_+^d).$$

This can be shown exactly the same fashion as [3, Theorem 2.2] using the simplified form of the Lévy-Kintchine formula (2.16). Then it is easily verified that $\mathcal{D} := \{x_f : x \in X,\ f \in \mathcal{C}\} \subset \mathcal{D}(A_S)$, where $\mathcal{C} := W_0^{1,1}(\mathbb{R}_+^d) \cap C^1(\mathbb{R}_+^d)$, is $||\cdot||_{\bigcap_{i=1}^d \mathcal{D}(A_i)}$-dense in $\bigcap_{i=1}^d \mathcal{D}(A_i)$ and that (2.17) holds for $x_f \in \mathcal{D}$. Finally, Taylor's formula

$$T(v)x_f = x_f + \sum_{i=1}^d \int_0^1 T(tv) v_i A_i x_f\,dt,\ x_f \in \mathcal{D},\ v \in \mathbb{R}_+^d,$$

yields the estimate

$$\|(T(v)x_f - x_f)\| \le K_d(M_T + 1)\frac{|v|_2}{1 + |v|_2}\,||x_f||_{\bigcap_{i=1}^d \mathcal{D}(A_i)},$$

and the proof can be completed the same way as the proof of Theorem 2.12. □

3. Examples

3.1. Subordinating the d-parameter translation semigroup

Let X be any of the spaces $C_0(\mathbb{R}^d)$, $\mathrm{UCB}(\mathbb{R}^d)$ or $L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$.[3] Then with μ^t from (1.1) the semigroup

$$[S(t)f](x) = \int_{\mathbb{R}^d} f(x-y)\,\mu^t(dy),\ f \in X, \qquad (3.1)$$

is strongly continuous on X and its generator is given by

$$\begin{aligned}(\mathcal{M}_\psi f)(s) = &-c^2 f(s) - a \cdot \nabla f(s) + \tfrac{1}{2}\nabla \cdot Q \nabla f(s) \\ &+ \int_{y \ne 0} \left(f(s-y) - f(s) + \frac{\nabla f(s) \cdot y}{1 + |y|_2^2} \right) \phi(dy),\ f \in D,\end{aligned}$$

where $\mathcal{D} = \{f \in X :\ D^\alpha f \in X$ with multi-index $|\alpha| \le 2\}$ for $X = C_0(\mathbb{R}^d)$ or $X = \mathrm{UCB}(\mathbb{R}^d)$ and $\mathcal{D} = W^{2,p}(\mathbb{R}^d)$ for $X = L_p(\mathbb{R}^d)$. The statement for $X = C_0(\mathbb{R}^d)$ can be found in [22, Theorem 31.5] (for the variable coefficient version, see, for example, [21]) and now it is a corollary of Theorem 2.12 which is based on the

[3]Here, $\mathrm{UCB}(\mathbb{R}^d)$ denotes the uniformly bounded and uniformly continuous functions on $\mathbb{R}^d$ and $C_0(\mathbb{R}^d)$ the continuous functions on $\mathbb{R}^d$ with zero limit as $|x|_2 \to \infty$.

$L_1(\mathbb{R}^d)$-result [3, Theorem 2.2]. Thus, the semigroup S on the function space X inherits several useful properties of the corresponding semigroup on the space $L_1(R^d)$ as long as the d-parameter translation is strongly continuous on X.

3.2. Infinitely divisible processes on a d-dimensional torus

Let X be the space of continuous functions on a d-dimensional torus, i.e.,

$$X = C_\pi([0, 2\pi]^d)$$

with $f \in C_\pi([0, 2\pi]^d)$ if and only if f is continuous and periodic in each dimension (i.e. in each component the value at 2π has to agree with the value at zero). Take $A_i = -\frac{\partial}{\partial x_i}$ with

$$\mathcal{D}(A_i) = \left\{ f \in C_\pi([0, 2\pi]^d) : \frac{\partial}{\partial x_i} f \in C_\pi([0, 2\pi]^d) \right\}.$$

The d-parameter group is then given by

$$G(t_1, \ldots, t_d) f(x_1, \ldots, x_d) = f(x_1 - t_1, \ldots, x_d - t_d) \tag{3.2}$$

using the periodic extension of f. If S is positive with $\|S\| = 1$, i.e. S corresponds to an infinitely divisible process in $\mathbb{R}^d$, then G_S is the corresponding infinitely divisible process on the $d-$dimensional torus and Theorem 2.12 yields the generator formula for this semigroup. This provides a simple proof of the formula in [12] in this special case. Infinitely divisible processes on cylinders are defined analogously.

3.3. Inhomogeneous fractional derivatives

Consider $X = L_1(\mathbb{R}^2)$ and

$$A_x f := -\frac{\partial}{\partial x}\left(v_1(x) f(x, y)\right); A_y f := -\frac{\partial}{\partial y}\left(v_2(y) f(x, y)\right)$$

for continuously differentiable v_i with $v_i(x) > 0$ for all x and $\int_{-\infty}^0 1/v_i(x)\, dx = \int_0^\infty 1/v_i(x)\, dx = \infty$. It is easy to check that the two-parameter flow group is then given by

$$T(t_1, t_2) f(x, y) = f(h_1(x, t_1), h_2(y, t_2)) v_1(h_1(x, t_1)) v_2(h_2(y, t_2)) / (v_1(x) v_2(y)) \tag{3.3}$$

where $h_i(x, t)$ are implicitly defined via

$$\int_{h_i(x,t)}^x \frac{1}{v_i(s)}\, ds = t.$$

Suppose that S is defined by (1.1) where μ_S^t is a strictly operator stable probability measure on $\mathbb{R}^d$. Operator stable laws are infinitely divisible laws with nice scaling properties, see [14, 17]. For a certain range of the scaling parameters, it is shown in [18] that

$$\psi(k) = -\mu \cdot ik + \int_{\|\theta\|=1} \int_0^\infty \left(e^{-ik \cdot r^H \theta} - 1 + ik \cdot r^H \theta \right) \frac{dr}{r^2} \lambda(d\theta)$$

for some scaling matrix H and spectral measure λ. Operator stable semigroups are important in applications to physics [16], hydrology [23], and finance [18], partly because their generators involve multidimensional analogues of fractional derivatives. Subordinating the flow group leads to the generator formula, using substitution in (2.10),

$$A_S f = \mu_1 A_x f + \mu_2 A_y f + \int_{\|\theta\|=1} \int_0^\infty \left(G(r^H\theta) f - f - \langle r^H\theta, (A_x f, A_y f) \rangle \right) \frac{dr}{r^2} \lambda(d\theta)$$

for all $f \in \mathcal{D}(A_x^2) \cap \mathcal{D}(A_y^2) \cap \mathcal{D}(A_x A_y)$, generalising the fractional derivative operator defined in [19] for $v_1 = v_2 = 1$.

In particular, if $\mu = 0$ and $H = \begin{pmatrix} 1/\alpha & 0 \\ 0 & 1/\alpha \end{pmatrix}$ for some $1 < \alpha < 2$, using the same argument as in Lemma 7.3.8 in [17], the log-characteristic function reduces to

$$\psi(k) = -\Gamma(1-\alpha) \int_{\|\theta\|=1} \langle ik, \theta \rangle^\alpha \, \lambda(d\theta).$$

Hence we obtain that in this case, using the functional calculus for group generators developed in [2],

$$A_S f = -\Gamma(1-\alpha) \int_{\|\theta\|=1} (-\theta_1 A_x - \theta_2 A_y)^\alpha f \, \lambda(d\theta), \; f \in \mathcal{D}(A_x^2) \cap \mathcal{D}(A_y^2) \cap \mathcal{D}(A_x A_y).$$

Similarly, if λ is concentrated on the axes and $H = \begin{pmatrix} 1/\alpha_1 & 0 \\ 0 & 1/\alpha_2 \end{pmatrix}$ we obtain in the simplest case

$$A_S f = -|A_x|^{\alpha_1} f - |A_y|^{\alpha_2} f.$$

For example, if $v_1 = v_2 \equiv 1$, then the generator A_S is a mixture of one variable fractional derivatives.

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