Operator Lévy motion and multiscaling anomalous diffusion

Mark M. Meerschaert†
Department of Mathematics, University of Nevada, Reno, 89557–0084

David A. Benson‡
Desert Research Institute, P.O. Box 60220, Reno, NV 89506-0220

Boris Baeumer§
Department of Geological Sciences, University of Nevada, Reno, 89557–0172

The long term, limit motions of individual heavy–tailed (power–law) particle jumps that characterize anomalous diffusion may have different scaling rates in different directions. Operator stable motions $\{Y(t) : t \geq 0\}$ are limits of $d-$dimensional random jumps that are scale–invariant according to $c^H Y(t) = Y(ct)$, where H is a $d \times d$ matrix. The eigenvalues of the matrix have real parts $1/\alpha_j$, with each positive $\alpha_j \leq 2$. In each of the j principle directions, the random motion has a different Fickian or super–Fickian diffusion (dispersion) rate proportional to t^{1/α_j}. These motions have a governing equation with a spatial dispersion operator that is a mixture of fractional derivatives of different order in different directions. Subsets of the generalized fractional operator include: (1) a fractional Laplacian with a single order α and a general directional mixing measure M; and (2) a fractional Laplacian with uniform mixing measure (the Riesz potential). The motivation for the generalized dispersion is the observation that tracers in natural aquifers scale at different (super–Fickian) rates in the directions parallel and perpendicular to mean flow.

PACS numbers: 05.40.+j, 05.60.+w, 47.55.Mh, 02.50.Cw

I. INTRODUCTION

Anomalous diffusion is an important process in hydrogeology because of the way that dissolved and often toxic chemical tracers move through aquifer material. Groundwater velocities span many orders of magnitude and give rise to diffusion–like dispersion (a term that combines molecular diffusion and hydrodynamic dispersion). The measured variance growth in the direction of flow of tracer plumes is typically at a super–Fickian rate, *i.e.*, $<X^2> \sim t^\gamma$, with $\gamma > 1$ [1, 2]. However, the scaling exponent γ is lower in the directions transverse to mean flow. Theories developed for anomalous dispersion in many dimensions are based on a coupling of space and time that gives a power γ that is equal in all directions. A prefactor (usually in the form of an effective dispersion coefficient) or mixing measure is used to make the process anisotropic [3–5].

The term "anomalous diffusion" has been defined several ways. A standard description requires that a particle in a spreading tracer cloud has a second moment that grows at a super–Fickian rate. This definition is restricted to processes with finite variance, so we choose a definition in terms of $P(x,t)$, the Green function of a diffusive–type equation or the probability density of a particle starting from the origin (the propagator). Anomalous diffusion has a space–time scaling property according to $P(x, ct) \sim c^{-1/\alpha} P(c^{-1/\alpha} x, t)$, which leads to a measured variance which grows proportional to $t^{2/\alpha}$. We consider super–Fickian and Fickian diffusion rates in which $0 < \alpha \leq 2$. The value of α should depend on direction in $d-$dimensional space. First we consider Brownian motion, Lévy motion, and their governing equations.

II. CLASSICAL ADVECTION AND DIFFUSION

The classical advection–dispersion equation models transport in porous media [6]. Let $X(t)$ be the position of a random tracer particle in d–dimensional Euclidean space $\mathbb{R}^d$ at time $t \geq 0$ and let $P(x,t)$ denote the probability density of $X(t)$, realized as an ensemble average, where the column vector $x \in \mathbb{R}^d$. Then

$$\frac{\partial P(x,t)}{\partial t} = (-v \cdot \nabla + \nabla \cdot A \nabla) P(x,t) \qquad (1)$$

---

†partially supported by NSF–DES grant 9980484; Electronic address: `mcubed@unr.edu`

‡partially supported by NSF–DES grant 9980489 and DOE–BES grant DE–FG03–98ER14885

§partially supported by NSF–DES grant 9980484

where $v \in \mathbb{R}^d$ is the average advective velocity, $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_d)^T$, and A is the $d \times d$ dispersion matrix. Take Fourier transforms $\hat{f}(k) = \int e^{-ik\cdot x} f(x)dx$ to obtain

$$\frac{d\hat{P}(k,t)}{dt} = \left(-v \cdot (ik) + (ik) \cdot A(ik)\right) \hat{P}(k,t)$$

whose solution $\hat{P}(k,t) = \exp\left(-itv \cdot k - tk \cdot Ak\right)$, corresponding to the point source initial condition $P(x,0) = \delta(x)$, inverts to the probability density of a multivariate normal random vector $X(t)$ on $\mathbb{R}^d$ with mean tv and covariance matrix $2tA$. If $A = I$ the identity matrix then the tracer plume is symmetric about its mean. If we write $X(t) = tv + Y(t)$ then the pure dispersion/diffusion process $Y(t)$ is normal with mean zero and covariance matrix $2tA$.

The stochastic process $Y(t)$ is the scaling limit of a random walk. If $W(t) = J_1 + \cdots + J_{[t]}$ where $J, J_1, J_2, J_3, \ldots$ are independent and identically distributed jumps with mean $EJ = 0$ and covariance matrix $EJJ^T = A$, the central limit theorem implies that $c^{-1/2}W(ct) \Rightarrow Y(t)$ as $c \to \infty$ for every $t \geq 0$, where $\Rightarrow$ means convergence of probability distributions. Physically this means that, if particle jumps relative to the center of mass of the plume resemble a random walk, then after sufficient time the distribution of an ensemble of particles will obey a normal concentration profile. For independent, identically distributed particle jumps with a finite covariance matrix, this multivariable Brownian motion $Y(t)$ is the only possible limiting process. In this sense, the classical advection–dispersion equation is a very robust large–scale model for contaminant transport.

In the classical advection–dispersion model, the center of mass of the contaminant plume is located at $x = tv$ and the mean–centered random particle location $Y(t)$ is a self–similar process with Hurst index $H = 1/2$, meaning that $Y(ct) = c^H Y(t)$ in distribution for any $t \geq 0$ and any scale $c > 0$. This plume spreads away from the center of mass like $t^{1/2}$ in every radial direction.

III. ANOMALOUS DISPERSION

In real world tracer tests [2, 7, 8] one typically observes anomalous dispersion, where the plume spreads faster than the classical model predicts. One alternative model involves correlated particle jumps, which can lead to fractional Brownian motion [9]. A mathematically simpler alternative is to relax the assumption that the covariance matrix exists. This case requires application of a generalized central limit theorem [10, 11]. If for some $0 < \alpha \leq 2$ we have

$$n^{-1/\alpha}(J_1 + \cdots + J_n) \Rightarrow Y \tag{2}$$

then the random particle jumps belong to the domain of attraction of Y, a stable random vector with index α. If $\alpha = 2$ then Y is normal, otherwise $P(\|Y\| > r) \sim Cr^{-\alpha}$ so that $E\|Y\|^2 = \infty$ and the covariance matrix of Y does not exist [12]. From (2) it follows easily that $c^{-H}W(ct) \Rightarrow Y(t)$ where $H = 1/\alpha$, and $\{Y(t) : t \geq 0\}$ is a multivariable Lévy motion, a stationary independent increment process with $Y = Y(1)$ stable with index α. Then $Y(ct) = c^H Y(t)$ so that H is the Hurst index of the limit process. Since $\alpha < 2$ we have $H > 1/2$ and the plume spreads out like t^H, faster then the classical model. Sample paths of $Y(t)$ are random fractals of dimension α, so the stable index also has physical meaning [13].

If we include linear advection then $X(t) = tv + Y(t)$ is a stable random vector whose density usually cannot be expressed in closed form. The Fourier transform of the density $P(x,t)$ can be written in terms of the Lévy representation

$$\hat{P}(k,t) = \exp\Big(-itv \cdot k - tk \cdot Ak + t\int \Big(e^{-ik\cdot x} - 1 + \frac{ik \cdot x}{1 + \|x\|^2}\Big)\phi(dx)\Big) \tag{3}$$

where v is a vector, A is a matrix, and ϕ is a Lévy measure, a Borel measure on $\mathbb{R}^d \setminus \{0\}$ such that the integral condition $\int \min\{1, \|x\|^2\}\phi(dx) < \infty$ holds [14]. If $v = 0$ then $Y(ct) = c^H Y(t)$ implies that $\hat{P}(k, ct) = \hat{P}(c^H k, t)$, and then (3) yields $ck \cdot Ak = c^{2H} k \cdot Ak$ and $c\phi(dx) = \phi(c^{-H}dx)$ for every scale $c > 0$. The scaling of the A term requires $H = 1/2$, and the scaling of ϕ requires $H > 1/2$ in view of the integral condition. If $H = 1/2$ then $X(t)$ is normal, ϕ is the zero measure, and the formula reduces to classical advection–dispersion. If $H > 1/2$ then A is the zero matrix and $\phi\{x : \|x\| > r, \frac{x}{\|x\|} \in S\} = Cr^{-\alpha}M(S)$ where M is a probability measure on the unit sphere in $\mathbb{R}^d$ called the mixing measure. For the central limit theorem (2) to hold with $\alpha < 2$ it suffices that $P(J \in dx) \sim \phi(dx)$ for $\|x\|$ large, so that the Lévy measure regulates the probability distribution of large jumps. The mean–centered process $Y(t)$ is the scaling limit of a random walk composed of these jumps, so that the advected process $X(t)$ consists of a linear drift plus a term which is essentially the accumulation of a large number of random jumps, where the probability of a large jump falls off like $r^{-\alpha}$ and $M(S)$ is the probability of jumping through a sector S of the unit sphere.

If $\alpha \neq 1$ the density $P(x,t)$ of the random particle location $X(t)$ solves a point source fractional advection–dispersion equation

$$\frac{\partial P(x,t)}{\partial t} = \left(-v \cdot \nabla + D\nabla_M^\alpha\right) P(x,t) \tag{4}$$

where $D = C\alpha\Gamma(-\alpha)$ and ∇_M^α is a mixture of fractional directional derivatives [5]. For any unit vector θ the fractional directional derivative of order α is the inverse Fourier transform of $(ik \cdot \theta)^\alpha \hat{f}(k)$ so $\nabla_M^\alpha f(x)$ is the inverse Fourier transform of

$$\left(\int_{\|\theta\|=1}(ik\cdot\theta)^\alpha M(d\theta)\right)\hat{f}(k). \quad (5)$$

IV. GENERALIZED DISPERSION

In anomalous dispersion a random particle location $X(t) = tv + Y(t)$ where v is the average advective velocity and the mean–centered dispersion process satisfies anomalous scaling $Y(ct) = c^H Y(t)$ for some $H > 1/2$. This implies that the tracer plume spreads like t^H in every coordinate direction. In real world tracer tests it is commonly observed that the rate of spreading depends on the coordinate [7, 8] which requires a generalized dispersion model. Let H represent an arbitrary $d\times d$ matrix and $t^H = \exp(H\log t)$ where $\exp(A) = I + A + A^2/2! + A^3/3! + \cdots$ is the usual exponential of a matrix. Since $\exp(tA)$ appears in the general solution to the linear differential equation $x' = Ax$, the behavior of the matrix power t^H can be obtained by a simple reparameterization [15]. For example, if $H = \mathrm{diag}(h_1,\ldots,h_d)$ a diagonal matrix then $t^H = \mathrm{diag}(t^{h_1},\ldots,t^{h_d})$. A stochastic process which satisfies $Y(ct) = t^H Y(t)$ for a matrix H is called operator self–similar [16].

If for some matrix H we have

$$n^{-H}(J_1 + \cdots + J_N) \Rightarrow Y \quad (6)$$

then the random particle jumps belong to the generalized domain of attraction [17–19] of some operator stable random vector Y with exponent H [20, 21]. If $H = \mathrm{diag}(h_1,\ldots,h_d)$ with $h_j = 1/\alpha_j$ then the jth component Y_j is stable with index α_j, and this requires $h_j \geq 1/2$ so that $0 < \alpha_j \leq 2$. Matrix norming in (6) allows the stable index α_j to vary with $j = 1,\ldots,d$. The physical meaning of the Hurst exponent H is that the jth component of the tracer plume spreads out from the center of mass like t^{1/α_j}.

The matrix H need not be diagonal. Eigenvalues of H are all of the form $1/\alpha + i\beta$ for some $0 < \alpha \leq 2$, where the imaginary part induces a rotation. A spectral decomposition [22, 23] resolves Y into d components in a coordinate system which depends on H. If $\alpha_j = 2$ then Y_j is normal, otherwise $\alpha_j < 2$ and the probability tails of Y_j fall off like $r^{-\alpha_j}$. The components for a given value of α_j span the generalized eigenspaces corresponding to eigenvalues with real part $1/\alpha_j$. From (6) it follows easily that $c^{-H}W(ct) \Rightarrow Y(t)$ where $\{Y(t) : t \geq 0\}$ is an operator Lévy motion, a stationary independent increment process with $Y = Y(1)$ operator stable with exponent H. Then $Y(ct) = c^H Y(t)$ so that $\{Y(t) : t \geq 0\}$ is operator self–similar.

If we include linear advection then $X(t) = tv + Y(t)$ is an operator stable random vector, and the Fourier transform of its density $P(x,t)$ can be written in terms of the Lévy representation (3). If ϕ is the zero measure then the formula reduces to classical advection–dispersion and $X(t)$ is normal Brownian motion with drift. The Lévy measure satisfies $c\phi(dx) = \phi(c^{-H}dx)$ for every scale $c > 0$, and the mathematical condition for the central limit theorem (6) to hold is that $P(J \in dx) \sim \phi(dx)$ for $\|x\|$ large, so the probability tails of J fall off like $r^{-\alpha_j}$ depending on direction. We can also mix normal and heavy tail components, which are evidently independent in view of (3) which expresses the two components as a product in Fourier space. In this case, scaling considerations show that the Lévy measure ϕ is supported on the subspace $\{k : k\cdot Ak = 0\}$, so that some coordinates of the underlying random walk have jumps with finite variance, and the rest have power law probability tails.

V. GENERALIZED ADVECTION–DISPERSION EQUATION

If $\alpha_j > 1$ for every j the Lévy representation (3) takes an alternate form

$$\hat{P}(k,t) = \exp\Big(-ita\cdot k - tk\cdot Ak + t\int(e^{-ik\cdot x} - 1 + ik\cdot x)\phi(dx)\Big) \quad (7)$$

where $ta = EX(t)$. Invert (7) to see that $P(x,t)$ solves a point source generalized advection–dispersion equation

$$\frac{\partial P(x,t)}{\partial t} = \left(-a\cdot\nabla + \nabla\cdot A\nabla + \mathcal{F}\right)P(x,t) \quad (8)$$

where the generalized fractional derivative $\mathcal{F}f(x)$ is the inverse Fourier transform of

$$\widehat{\mathcal{F}f}(k) = \left(\int(e^{-ix\cdot k} - 1 + ik\cdot x)\phi(dx)\right)\hat{f}(k). \quad (9)$$

Distribute $\hat{f}(k)$ and invert inside the integral (9) to get

$$\mathcal{F}f(x) = \int\left(f(x-y) - f(x) + y\cdot\nabla f(x)\right)\phi(dy) \quad (10)$$

which is essentially convolution with the Lévy measure. If $1 < \alpha < 2$ the fractional derivative [24]

$$\frac{d^\alpha}{dx^\alpha}f(x) = \frac{1}{\Gamma(2-\alpha)}\int_0^\infty f''(x-y)y^{1-\alpha}dy \quad (11)$$

and if $\phi(dy) = C\alpha r^{-\alpha-1}dy$ on $y > 0$ integration by parts in (10) yields

$$\begin{aligned}\mathcal{F}f(x) &= C\alpha\int_0^\infty\left(f(x-y) - f(x) + yf'(x)\right)y^{-\alpha-1}dy\\ &= C\int_0^\infty\left(-f'(x-y) + f'(x)\right)y^{-\alpha}dy\\ &= \frac{C}{(1-\alpha)}\int_0^\infty f''(x-y)y^{1-\alpha}dy\end{aligned}$$

so that $\mathcal{F} = d^\alpha/dx^\alpha$ if $C = (1-\alpha)/\Gamma(2-\alpha)$. In this case $X(t)$ is a maximally skewed Lévy motion with drift [25, 26]. If $\phi(dy) = pC\alpha r^{-\alpha-1}dy$ on $y > 0$ and $\phi(dy) = qC\alpha r^{-\alpha-1}dy$ on $y < 0$ then $\mathcal{F} = pd^\alpha/dx^\alpha + qd^\alpha/d(-x)^\alpha$ and we recover every Lévy motion with drift [27, 28]. If $\phi(dx) = C\alpha r^{-\alpha-1}drM(d\theta)$ in polar coordinates then

$$\mathcal{F}f(x) = \int_{\|\theta\|=1} \frac{d^\alpha}{dr^\alpha} f(x - r\theta)M(d\theta)$$

a mixture of fractional directional derivatives. Then $\mathcal{F} = \nabla_M^\alpha$ and we recover every multivariable Lévy motion with drift [5]. If M is symmetric then $\mathcal{F} = \cos(\pi\alpha/2)\Delta^{\alpha/2}$ where $\Delta^{\alpha/2}$ is the classical Riesz fractional derivative, and in this case the density $P(x,t)$ of the Lévy motion $X(t)$ is spherically symmetric about its mean.

The fractional derivative (11) is a convolution, so its Fourier transform is a product. Standard tables [29] give $\hat{g}(k) = (ik)^q$ as the Fourier transform of $g(x) = H(x)x^{-q-1}/\Gamma(-q)$ for $-1 < q < 0$, where $H(x)$ is the Heaviside function. Since $f''(x)$ has Fourier transform $(ik)^2\hat{f}(k)$ the fractional derivative $d^\alpha f(x)/dx^\alpha$ has Fourier transform $(ik)^2\hat{f}(k)(ik)^{\alpha-2} = (ik)^\alpha\hat{f}(k)$ which generalizes the familiar integer derivative formula. Since $\hat{g}(k) = \int e^{-ikx}g(x)dx$ is an improper integral which diverges at infinity for any $-1 < q < 0$, the meaning of the table entry is that the convolution $g * h(x) = \int h(x-y)g(y)dy$ has Fourier transform $(ik)^q\hat{h}(k)$ for smooth functions $h(x)$ which vanish off a compact set.

Formal integration by parts in (11) yields

$$\frac{d^\alpha}{dx^\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty f(x-y)y^{-\alpha-1}dy \qquad (12)$$

a convolution of generalized functions [29], so that the fractional derivative

$$\mathcal{F}f(x) = \frac{d^\alpha}{dx^\alpha} f(x) = \int f(x-y)\phi(dy) \qquad (13)$$

is convolution with the Lévy measure $\phi(dy) = y^{-\alpha-1}dy/\Gamma(-\alpha)$ on $y > 0$ in this distributional sense. If $\phi(dx) = r^{-\alpha-1}M(d\theta)/\Gamma(-\alpha)$ a similar argument yields

$$\mathcal{F}f(x) = \nabla_M^\alpha f(x) = \int f(x-y)\phi(dy) \qquad (14)$$

a mixture of integrals involving the generalized function $r^{-\alpha-1}/\Gamma(-\alpha)$. We conjecture that the generalized fractional derivative operator is always equivalent to convolution with the Lévy measure $\mathcal{F}f(x) = \int f(x-y)\phi(dy)$, but this is an open question whose resolution seems to require a suitable extension of generalized function theory.

VI. DISCUSSION

Generalized dispersion is an attractive model because it is based on an operator limit theorem. Heavy–tailed (power–law) jumps in d dimensions will converge to the operator stable laws governed by Equation (3). If independent thin–tailed jumps are mixed with heavy–tailed ones, the index of the heaviest jumps dominates. Granular aquifer material is generally deposited in sheet–like to tube–like structures of similar grain size [30]. Fractures in crystalline rock will have preferred directions and lengths [31] according to the external stress field. Many of the transport properties (aperture, displacement, length, connectivity) of natural fractures and faults have power-law distributions and scaling [32–34].

Several tracer tests have been conducted with sufficient sampling detail to resolve anomalous dispersion [7, 8]. Two tests show significant differences in the measured variance growth rate in the longitudinal and transverse directions [Figure 1]. In both of the tests, the vertical growth rate was essentially zero, so that the plume growth is two–dimensional. Longitudinal plume dispersion in the relatively homogeneous Cape Cod aquifer is anomalous with $\alpha \approx 1.65$ and Fickian lateral dispersion [Figure 1 (a)]. Here the generalized advection–dispersion

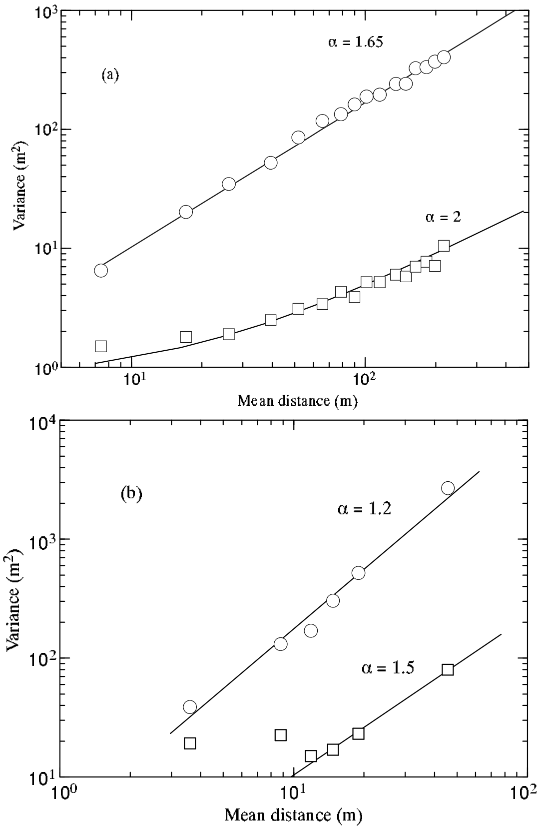

FIG. 1: Measured longitudinal (circles) and lateral (squares) variance of the bromide plumes versus mean travel distance in the (a) Cape Cod [7] and (b) MADE site [8] aquifers. Lines indicate power laws of order $2/\alpha$. Transverse values are artifically high at early time due to the wide (~ 5m) arrays of injection wells.

model resolves into two independent processes, a longitudinal Lévy motion with index α and a lateral Brownian motion. At the highly heterogeneous MADE site, both longitudinal and lateral dispersion are anomalous, but with different scaling [Figure 1 (b)]. The longitudinal $\alpha_1 \approx 1.2$ and the lateral $\alpha_2 \approx 1.5$ so the generalized advection–dispersion model involves a longitudinal Lévy motion with index α_1 and a lateral Lévy motion with index α_2. The heaviest tail of the MADE plume corresponds to the smallest index α_1, and this value is consistent with the observed tail index ($\alpha \approx 1.1$) of the hydraulic conductivity (K) random field [1]. An assumption that the hydraulic head gradient was fairly uniform in the longitudinal direction led to a scalar fractional advection dispersion equation of order 1.1. It may also be possible to obtain *a priori* estimates of the lateral index α_2 based on a model of the K field which recognizes the possibility of operator scaling.

VII. CONCLUSIONS

Stable random processes are the limit sums of random jumps in many dimensions. If the scaling properties are the same in all directions, then the process is governed by a multidimensional advection–dispersion equation. The fractional–order Laplacian is a mixture of any number of fractional directional derivatives, with weights defined by a mixing measure $M(d\theta)$ that can be discrete or continuous. This operation is equivalent to convolution with a Lévy measure whose tails diminish like $r^{-\alpha}$ in every direction. If the scaling is different in each dimension then the fractional dispersion operator applies a fractional derivative of different order in each coordinate. The operator stable motions may contain independent Brownian motion, which can be overwhelmed by any additional heavy–tailed jumps with components in the same direction. If no heavy–tailed jumps occur in a particular direction, then the scaling rate of Brownian motion (with $t^{1/2}$) prevails. The detailed plume studies at the Cape Cod and MADE site aquifers show evidence of generalized dispersion.

REFERENCES

[1] D. Benson et al., in press, Trans. Porous Media.
[2] D. Benson, S. Wheatcraft and M. Meerschaert, Water Resour. Res. **36**, 1403 (2000).
[3] H. Rajaram and L. W. Gelhar, Water Resour. Res. **31**, 2469 (1995).
[4] H. Zhan and S. W. Wheatcraft, Water Resour. Res. **32**, 3461 (1996).
[5] M. Meerschaert, D. Benson and B. Baeumer, Phys. Rev. E **59**, 5026 (1999).
[6] J. Bear, *Dynamics of Fluids in Porous Media* (Dover Publications, NY, 1972).
[7] S. Garabedian *et. al.* , Water Resour. Res. **27**, 911 (1991).
[8] E. Adams and L. Gelhar, Water Resour. Res. **28**, 3293 (1992).
[9] F. Molz, H. Liu and J. Szulga, Water Resour. Res. **33** 2273 (1997).
[10] E. Rvačeva, *Select. Transl. Math. Stat. Prob.* Amer. Math. Soc. **2**, 183 (1962).
[11] M. Meerschaert, Stat. Probab. Lett. **4**, 43 (1986).
[12] G. Samorodnitsky and M. Taqqu, *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance* (Chapman and Hall, London, 1994).
[13] S. Taylor, Math. Proc. Cambridge Philos. Soc. **100**, 383 (1986).
[14] M. Meerschaert and H.P. Scheffler, *Limit Theorems for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice* (Wiley, New York, 2000).
[15] M. Hirsch and S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra* (Academic Press, New York, 1974).
[16] M. Maejima and J.D. Mason, Stoch. Proc. Appl. **54**, 139 (1994).
[17] Z. Jurek, Bull. Acad. Pol. Sci. **28**, 397 (1980).
[18] M. Hahn and M. Klass, Z. Wahrsch. verw. Geb. **69**, 479 (1985).
[19] M. Meerschaert, Statist. Probab. Lett. **18**, 233 (1993).
[20] M. Sharpe, Trans. Amer. Math. Soc. **136**, 51 (1969).
[21] Z. Jurek and J.D. Mason, *Operator–Limit Distributions in Probability Theory* (Wiley, New York, 1993).
[22] M. Meerschaert, Ann. Probab. **19**, 875 (1991).
[23] M. Meerschaert and H.P. Scheffler, Stoch. Proc. Appl. **84**, 71 (1999).
[24] S. Samko, A. Kilbas and O. Marichev *Fractional Integrals and derivatives: Theory and Applications* (Gordon and Breach, London, 1993).
[25] G. M. Zaslavsky, Physica D **76**, 110 (1994).
[26] A. Compte, Phys. Rev. E **55**, , 6821 (1997).
[27] A. S. Chaves, Phys. Lett. A **239**, 13 (1998).
[28] D. Benson, S. Wheatcraft and M. Meerschaert, Water Resour. Res. **36**, 1413 (2000).
[29] A. Zemanian, *Distribution Theory and Transform Analysis: An Introduction to Generalized Functions, with Applications* (Dover, New York, 1987).
[30] Weissman, *et al.*, Water Resour. Res. **35**, 1761 (1999).
[31] For an overview, see National Research Council, *Rock Fractures and Fluid Flow* (National Academy Press, Washington, D.C., 1996).
[32] R. Marrett, *et al.*, Geology **27**, 799 (1999).
[33] P. Segall and D. Pollard, Geol. Soc. Amer. Bull. **94**, 563 (1983).
[34] O. Bour and P. Davy, Water Resour. Res. **33**, 1567 (1997).