STOCHASTIC SOLUTIONS FOR FRACTIONAL CAUCHY PROBLEMS

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Abstract

Every infinitely divisible law defines a convolution semigroup that solves an abstract Cauchy problem. In the fractional Cauchy problem, we replace the first order time derivative by a fractional derivative. Solutions to fractional Cauchy problems are obtained by subordinating the solution to the original Cauchy problem. Fractional Cauchy problems are useful in physics to model anomalous diffusion.

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1. Introduction

Infinitely divisible laws are weak limits of sums of independent and identically distributed random vectors. In statistical mechanics, these random vectors represent particle jumps, and the infinitely divisible law is the accumulation of arbitrarily many independent jumps. The associated Lévy process, a process with stationary independent increments distributed like some convolution power of the infinitely divisible law, represents the particle location at time $t \geq 0$. These convolution powers form a semigroup with generator L, and the application of this semigroup to some test function $g(x)$ solves the Cauchy problem $\partial C(x,t)/\partial t = LC(x,t); C(x,0) = g(x)$. For a standard multivariable normal law, L is the Laplacian operator $\sum_i \partial^2/\partial x_i^2$ and the Lévy process is a multivariable

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Brownian motion. For an (operator) stable law the generator involves fractional derivatives in space, and the Lévy process is an (operator) Lévy motion. These processes are the scaling limits of simple random walks, and are useful in physics to model superdiffusion, where particle concentration spreads faster than the classical Brownian motion predicts. Recently Saichev and Zaslavsky [18] introduced the fractional Cauchy equation $\partial^\beta C(x,t)/\partial t^\beta = LC(x,t); C(x,0) = g(x)$ where L is the generator of a stable semigroup in one dimension. Their solution is obtained via Bochner subordination from the solution to the classical Cauchy problem with the same generator. In this paper, we extend this result to an arbitrary infinitely divisible generator on a finite dimensional vector space. In the process, we provide rigorous mathematical proofs of some formulas which occur in the physics literature.

2. Cauchy Problems and Convolution Semigroups

Suppose that Y is a random vector on $\mathbf{R^d}$ with probability distribution μ and characteristic function ω. Let $\mu^n = \mu * \cdots * \mu$ denote the $n-$*fold convolution* of μ with itself. We say that Y (or μ or ω) is *infinitely divisible* if for each $n = 1, 2, 3, \ldots$ there exist $Y_{n1}, \ldots, Y_{nn}$ i.i.d. such that $Y_{n1} + \cdots + Y_{nn}$ is identically distributed with Y. Hence if Y_{ni} has distribution μ_n and characteristic function ω_n then $\mu_n^n = \mu$ and $\omega = \omega_n^n$. The *Lévy representation* (e.g., see Theorem 3.1.11 in [13]) states that a probability measure μ on $\mathbf{R^d}$ is infinitely divisible if and only if we can write the characteristic function $\hat{\mu}(k) = E[e^{ik\cdot Y}]$ in the form $\exp(\psi(k))$ where

$$\psi(k) = ik \cdot a - \frac{1}{2} k \cdot Ak + \int_{x \neq 0} \Big(e^{ik\cdot x} - 1 - \frac{ik \cdot x}{1 + \|x\|^2} \Big) \phi(dx), \tag{1}$$

where $a \in \mathbf{R^d}$, A is a nonnegative definite matrix, and ϕ is a σ-finite Borel measure on $\mathbf{R^d} \setminus \{0\}$ such that

$$\int_{x \neq 0} \min\{1, \|x\|^2\} \phi(dx) < \infty. \tag{2}$$

The triple $[a, A, \phi]$ is unique, and we call this the Lévy representation of the infinitely divisible law μ. A σ-finite Borel measure on $\mathbf{R^d} \setminus \{0\}$ such that (2) holds is called a *Lévy measure.* In this case, we define the convolution power μ^t to be the infinitely divisible law with Lévy representation $[ta, tA, t\phi]$, so that μ^t has characteristic function $\exp(t\psi(k))$ for any $t \geq 0$. Then it follows that $\mu^s * \mu^t = \mu^{s+t}$ for any $s, t \geq 0$.

Let $C_b(\mathbf{R^d})$ denote the class of all bounded, continuous functions $f : \mathbf{R^d} \to \mathbf{R^1}$. Given probability measures μ_n, μ on $\mathbf{R^d}$ we say that μ_n *converges weakly* to μ, and we write $\mu_n \Rightarrow \mu$, if $\int f(x)\mu_n(dx) \to \int f(x)\mu(dx)$ for all $f \in C_b(\mathbf{R^d})$. The Portmanteau theorem [see for example [13] Theorem 1.2.2] says that $\mu_n \Rightarrow \mu$

if and only if $\mu_n(B) \to \mu(B)$ for any Borel set $B \subseteq \mathbf{R^d}$ satisfying $\mu(\partial B) = 0$. A sequence of infinitely divisible laws (μ_n) on $\mathbf{R^d}$ with Lévy representations $[a_n, A_n, \phi_n]$ converges weakly to a probability measure μ on $\mathbf{R^d}$ if and only if:

(a) for some Lévy measure ϕ we have

$$\phi_n \to \phi \tag{3}$$

meaning that $\phi_n(B) \to \phi(B)$ for any Borel set $B \subseteq \mathbf{R^d} \setminus \{0\}$ such that $\phi(\partial B) = 0$;

(b) $a_n \to a$ for some $a \in \mathbf{R^d}$; and

(c) for some nonnegative definite matrix A we have

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \left[\int_{0<\|x\|<\varepsilon} (k \cdot x)^2 \phi_n(dx) + k \cdot A_n k \right] = k \cdot Ak \tag{4}$$

for all $k \in \mathbf{R^d}$.

In this case μ is infinitely divisible with Lévy representation $[a, A, \phi]$, see for example Theorem 3.1.16 in [13]. Then it follows that $\mu^{t_n} \Rightarrow \mu^t$ whenever $t_n, t \geq 0$ and $t_n \to t$.

Next we recall one of the main properties of infinitely divisible distribution, which is their connection to convolution semigroups. The proof of the proposition below is a straightforward adaption of Thm 23.13.1 in [10] to d dimensions. Let $\|f\|_1 = \int |f(x)| dx$ be the usual norm on the Banach space $L^1(\mathbf{R^d})$ of absolutely integrable functions $f : \mathbf{R^d} \to \mathbf{R^1}$. A family of linear operators $\{T(t) : t \geq 0\}$ on a Banach space X such that $T(0)$ is the identity operator and $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$ is called a *continuous convolution semigroup*. If $\|T(t)f\| \leq M\|f\|$ for all $f \in X$ and all $t \geq 0$ then the semigroup is *uniformly bounded*. If $T(t_n)f \to T(t)f$ in X for all $f \in X$ whenever $t_n \to t$ then the semigroup is *strongly continuous*. It is easy to check that $\{T(t) : t \geq 0\}$ is strongly continuous if $T(t)f \to f$ in X for all $f \in X$ as $t \to 0$. A strongly continuous convolution semigroup such that $\|T(t)f\| \leq \|f\|$ for all $t \geq 0$ and all $f \in X$ is also called a *Feller semigroup*. The next result is well known, but we could not find an accessible reference. It shows that any infinitely divisible law on $\mathbf{R^d}$ defines a Feller semigroup on $L^1(\mathbf{R^d})$.

PROPOSITION 2.1. *Let μ^t be infinitely divisible and define*

$$T(t)f(x) = \int f(x-s)\mu^t(ds) \tag{5}$$

for all $f \in L^1(\mathbf{R^d})$ and all $t \geq 0$. Then the family of linear operators $\{T(t)\}_{t \geq 0}$ has the following properties valid for all $f \in L^1(\mathbf{R^d})$:

(a) $T(t+s)f = T(t)T(s)f$ *for all* $t, s \geq 0$.

(b) $T(0)f = f$.

(c) $\|T(t)f\|_1 \leq \|f\|_1$ *for all* $t \geq 0$.

(d) $\lim_{t\to 0^+} \|T(t)f - f\|_1 \to 0$.

P r o o f. The points (a) and (b) follow immediately from the definition of μ^t being infinitely divisible. Point (c) follows from Fubini's theorem:

$$\|T(t)f\|_1 = \int \left| \int f(x-s)\mu^t(ds) \right| dx \leq \int \int |f(x-s)|\, dx\, \mu^t(ds) \leq \|f\|_1.$$

Point (d) is a bit more delicate. We first show that (d) holds for a characteristic function on a rectangle; i.e., let $f(x) = I_Q(x) = I(x \in Q)$ where the rectangle $Q = \{x \in \mathbf{R^d} : a \leq x \leq b\}$ for some $a, b \in \mathbf{R^d}$. Here we use the convention that $a \leq b$ if $a_i \leq b_i$ for all $1 \leq i \leq d$. Then

$$\begin{aligned} \int |T(t)f(x)| dx &= \int \left| \int f(x-s)\mu^t(ds) \right| dx \\ &= \int \int_{x-Q} \mu^t(ds)\, dx \\ &= \int \int_{s+Q} dx\, \mu^t(ds) \\ &= \prod_{1 \leq i \leq d} (b_i - a_i) \end{aligned} \tag{6}$$

for all $t \geq 0$. Furthermore, since μ^t is infinitely divisible we have that $\mu^t \Rightarrow \mu^0$ as $t \downarrow 0$, and therefore, for all $x \in \mathbf{R^d}$ with $x_i \neq a_i, b_i$, we obtain that

$$\lim_{t\to 0^+} T(t)f(x) = \lim_{t\to 0^+} \mu^t(x-Q) = \mu^0(x-Q) = f(x).$$

Thus, by the Dominated Convergence Theorem,

$$\int_Q |T(t)f(x) - f(x)|\, dx = \int_Q |\mu^t(x-Q) - \mu^0(x-Q)|\, dx \to 0$$

and thus, $\int_Q |T(t)f(x)|\, dx \to \int f(x)dx = \prod_{1\leq i \leq d}(b_i - a_i)$. In light of (6), this implies that

$$\int_{x \notin Q} |T(t)f(x) - f(x)|\, dx = \int_{x \notin Q} |T(t)f(x)|\, dx \to 0.$$

Hence, $\int |T(t)f(x) - f(x)| dx \to 0$ so that (d) holds for such f. Then it follows easily that (d) holds for any $f = \sum_{j=1}^n \alpha_j I_{Q_j}$. Now for any $f \in L^1(\mathbf{R^d})$ and any positive integer n it is well known that there exists a continuous $f_n \in L^1(\mathbf{R^d})$ with compact support and $\|f - f_n\|_1 \leq n^{-1}$, see for example Theorem 3.14 in [17]. Take a Riemann approximation to the integral $\int f_n(x)dx$ to obtain $g_n = \sum \alpha_j I_{Q_j}$ with $\|g_n - f_n\|_1 < n^{-1}$. Then it follows by the triangle inequality that (d) holds for all $f \in L^1(\mathbf{R^d})$.

For any strongly continuous semigroup $\{T(t) : t > 0\}$ on a Banach space X we define the *generator*

$$Lf = \lim_{t \to 0^+} \frac{T(t)f - f}{t} \quad \text{in } X \tag{7}$$

meaning that $\|t^{-1}(T(t)f - f) - Lf\| \to 0$ in the Banach space norm. The domain $D(L)$ of this linear operator is the set of all $f \in X$ for which the limit in (7) exists. Then $D(L)$ is dense in X, and L is closed, meaning that if $f_n \to f$ and $Lf_n \to g$ in X then $f \in D(L)$ and $Lf = g$ (see, for example, [16] Cor. I.2.5). In the following theorem we characterize the generator for the semigroup induced by an infinitely divisible law on $L^1(\mathbf{R^d})$. Recall that the Sobolev space $W^{2,1}(\mathbf{R^d})$ is the space of L^1-functions whose first and second partial derivatives are all L^1-functions. Furthermore note that we define the Fourier transform to be

$$\hat{f}(k) := \int e^{ik \cdot x} f(x)\, dx.$$

THEOREM 2.2. *Suppose that $T(t)$ is defined by (5) for all $f \in X = L^1(\mathbf{R^d})$, where μ is infinitely divisible with Lévy representation $[a, A, \phi]$, and let L be the generator of this strongly continuous semigroup. Then $\widehat{Lf}(k) = \psi(k)\hat{f}(k)$ for all $f \in D(L)$, where $\psi(k)$ is given by (1), and $D(L) = \{f \in L^1(\mathbf{R^d}) : \psi(k)\hat{f}(k) = \hat{h}(k)\ \exists\, h \in L^1(\mathbf{R^d})\}$. Furthermore, we have*

$$\begin{aligned} Lf(x) &= -a \cdot \nabla f(x) + \frac{1}{2} \nabla \cdot A \nabla f(x) \\ &\quad + \int_{y \neq 0} \Big(f(x-y) - f(x) + \frac{\nabla f(x) \cdot y}{1 + y^2} \Big) \phi(dy) \end{aligned} \tag{8}$$

for all $f \in W^{2,1}(\mathbf{R^d})$.

P r o o f. If $f \in D(L)$ then $Lf \in L^1(\mathbf{R^d})$, and hence the Fourier transform $\hat{f}(k) = \int e^{ik \cdot x} f(x)\, dx$ exists. Since the Fourier transform of a convolution is a product (e.g., see [13] Proposition 1.3.5), the Fourier transform $\widehat{T(t)}f(k) = \exp(t\psi(k))\hat{f}(k)$, where $\psi(k)$ is given by (1). Then since $\|\hat{f} - \hat{g}\|_1 \leq \|f - g\|_1$ for any $f, g \in L^1(\mathbf{R^d})$ it follows from (7) and the continuity of the Fourier transform that

$$\widehat{Lf}(k) = \lim_{t \to 0^+} \frac{\exp(t\psi(k)) - 1}{t} \hat{f}(k) = \psi(k)\hat{f}(k) \tag{9}$$

for all $f \in D(L)$.

Conversely, let $f \in L^1(\mathbf{R^d})$ be such that $\psi(k)\hat{f}(k) = \hat{h}(k)$ for some $h \in L^1(\mathbf{R^d})$. Then $g := \lambda f - h \in L^1(\mathbf{R^d})$ for all $\lambda > 0$. Furthermore, it is a basic fact of semigroup theory (see, e.g. [16] Thm. I.5.2) that the resolvent operator $(\lambda I - L)^{-1}$ is a bounded linear operator for all $\lambda > 0$ and maps $L^1(\mathbf{R^d})$ into $D(L)$. Let $q = (\lambda I - L)^{-1} g$. Then $\lambda q - Lq = g$ and $\lambda \hat{q}(k) - \psi(k)\hat{q}(k) = \hat{g}(k)$. Thus

$$\hat{q}(k) = \frac{\hat{g}(k)}{\lambda - \psi(k)} = \frac{\lambda \hat{f}(k) - \psi(k)\hat{f}(k)}{\lambda - \psi(k)} = \hat{f}(k).$$

Hence $q = f$ and therefore, $f \in D(L)$.

Using Taylor series expansion,

$$f(x-y) - f(x) + y \cdot \nabla f(x) = y \cdot \int_0^1 (1-t) M_{x-ty} y \, dt,$$

where M_x is the Hessian matrix of f evaluated at x, it is easy to check that

$$\int |f(x-y) - f(x) + \frac{y \cdot \nabla f(x)}{1 + \|y\|^2}| \, dx \le C \|f\|_{W^{2,1}} \frac{\|y\|^2}{1 + \|y\|^2}.$$

Using the Fubini theorem, we see that (8) is well defined for all $f \in W^{2,1}(\mathbf{R^d})$. Since $(-ik_j)\hat{f}(k)$ is the Fourier transform of $\partial f(x)/\partial x_j$, and $\exp(ik \cdot y)\hat{f}(k)$ is the Fourier transform of $f(x-y)$, it follows that the right hand side of (8) has Fourier transform $\psi(k)\hat{f}(k)$.

Another consequence of $T(t)$ being a strongly continuous semigroup is that $u(t) = T(t)f$ solves the abstract *Cauchy problem*

$$\frac{d}{dt} u(t) = Lu(t); \quad u(0) = f \tag{10}$$

for $f \in D(L)$. Furthermore, the integrated equation $T(t)f = L \int_0^t T(s) f \, ds + f$ holds for all $f \in X$ (see, for example, [16] Theorem I.2.4.). More specifically, for $X = L^1(\mathbf{R^d})$, define $q(x,t) = \int p_0(x-y) \mu^t(dy) = T(t)p_0(x)$, where $p_0 \in D(L)$ is a given initial condition. Then

$$\frac{\partial q(x,t)}{\partial t} = Lq(x,t); \quad q(x,0) = p_0(x) \tag{11}$$

for all $t > 0$ and $x \in \mathbf{R^d}$. If μ^t has probability density $p(x,t)$ for $t > 0$, then $q(x,t) = \int p_0(x-y) p(y,t) dy$ and $\{p(x,t) : t > 0\}$ is called the *Green's function solution* to this Cauchy problem.

Stable laws on $\mathbf{R^1}$ are a special case of infinitely divisible laws. Suppose that Y is a random variable with probability distribution μ and characteristic function ω. We say that Y (or μ or ω) is *stable* with index α if for each $n = 1, 2, 3, \ldots$ and $Y_1, \ldots, Y_n$ i.i.d. with Y we have

$$a_n^{-1}(Y_1 + \cdots + Y_n) - b_n \stackrel{d}{=} Y \tag{12}$$

for some constants $a_n > 0$ and $b_n \in \mathbf{R^1}$. In terms of characteristic functions this means that then $\omega(a_n^{-1}k)^n \exp(-ikb_n) = \omega(k)$ for all $k \in \mathbf{R^1}$. A normal law μ with mean a and variance σ^2 has characteristic function $\omega(k) = \exp(ika - \sigma^2 k^2/2)$ and it is easy to check that (12) holds with $a_n = n^{1/2}$. Any other stable law has Lévy representation $[a, 0, \phi]$ where ϕ is of the form

$$\phi(r, \infty) = pCr^{-\alpha} \quad \text{and} \quad \phi(-\infty, -r) = qCr^{-\alpha} \tag{13}$$

for every $r > 0$, for some $C > 0$ and $p \geq 0$, $q \geq 0$ with $p + q = 1$, and some $0 < \alpha < 2$, see for example Corollary 7.3.4 in [13]. If $\alpha \neq 1$ then μ has characteristic function $\exp(\psi(t))$ where

$$\psi(t) = ia_1 t - \sigma^\alpha |t|^\alpha \left(1 - i\beta \mathrm{sign(t)} \tan\left(\frac{\pi\alpha}{2}\right)\right) \tag{14}$$

for some $a_1 \in \mathbf{R^1}$ with $\beta = p - q$ and

$$\sigma^\alpha = C \cdot \frac{\Gamma(2-\alpha)}{1-\alpha} \cdot \cos\left(\frac{\pi\alpha}{2}\right), \tag{15}$$

see for example Theorem 7.3.5 of [13]. Then it is easy to verify that (12) holds with $a_n = n^{1/\alpha}$.

For any infinitely divisible law μ, we can define the *Lévy process* $\{A(t)\}_{t\geq 0}$, a stochastic process with stationary, independent increments such that μ^t is the probability distribution of $A(t)$. If μ^t has density $p(x, t)$ it follows that $p(x, t)$ is the Green's function solution to the Cauchy problem (11), where L is the generator of the Feller semigroup $\{T(t)\}$ defined by (5). In this case, we call the Lévy process $\{A(t)\}$ a *stochastic solution* to the Cauchy problem (11). If μ is stable then we call the associated Lévy process a stable *Lévy motion*. In this case, the generator L can be written in terms of *fractional derivatives*.

The *fractional derivative* is defined for $0 < \alpha < 1$ as

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty (f(x-y) - f(x))\alpha y^{-1-\alpha} dy$$

and for $1 < \alpha < 2$ as

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{\alpha - 1}{\Gamma(2-\alpha)} \int_0^\infty (f(x-y) - f(x) + yf'(x))\alpha y^{-1-\alpha} dy.$$

The negative fractional derivative $d^\alpha f(x)/d(-x)^\alpha = d^\alpha g(-x)/dx^\alpha$ where $g(x) = f(-x)$.

THEOREM 2.3. *Suppose μ is stable with Lévy representation $[a, 0, \phi]$ where ϕ is of the form (13), and L is the generator of the Feller semigroup $\{T(t)\}$ defined by (5). Then for any $0 < \alpha < 1$ or $1 < \alpha < 2$ we have for all $f \in W^{2,1}(\mathbf{R^1})$ that*

$$Lf(x) = -a_2 \frac{df(x)}{dx} + Dq \frac{d^\alpha f(x)}{d(-x)^\alpha} + Dp \frac{d^\alpha f(x)}{dx^\alpha} \tag{16}$$

for some $a_2 \in \mathbf{R^1}$*, where* C, p, q *are as in (13) and* $D = C\Gamma(2-\alpha)/(\alpha-1)$.

P r o o f. Theorem 2.2 implies that $\widehat{Lf}(k) = \psi(k)\hat{f}(k)$ for all $f \in W^{2,1}(\mathbf{R^1})$, where

$$\begin{aligned}\psi(k) = ika \quad &+ \quad Cp\int_0^\infty \left(e^{iky} - 1 - \frac{iky}{1+y^2}\right)\alpha y^{-\alpha-1}dy \\ &+ \quad Cq\int_0^\infty \left(e^{-iky} - 1 + \frac{iky}{1+y^2}\right)\alpha y^{-\alpha-1}dy \qquad (17)\end{aligned}$$

in view of (1) and (13). First suppose that $0 < \alpha < 1$. An elementary computation (see for example Lemma 7.3.7 in [13]) shows that

$$\int_0^\infty (e^{iky} - 1)\alpha y^{-\alpha-1}dy = -\Gamma(1-\alpha)(-ik)^\alpha$$

and since $0 < \alpha < 1$ the integral

$$b = \int_0^\infty \frac{y}{1+y^2}\alpha y^{-\alpha-1}dy$$

converges. Then we can write

$$\psi(k) = -ika_2 + Dq(ik)^\alpha + Dp(-ik)^\alpha \qquad (18)$$

where C, p, q are as in (13), $D = C\Gamma(2-\alpha)/(\alpha-1)$ and $a_2 = C(p-q)b - a$. If $f \in W^{2,1}(\mathbf{R^1})$ then $d^\alpha f(x)/dx^\alpha \in L^1(\mathbf{R^d})$ and it follows using the Fubini theorem that

$$\begin{aligned}\int e^{ikx}\frac{d^\alpha f(x)}{dx^\alpha}dx &= \frac{-1}{\Gamma(1-\alpha)}\int_0^\infty \left(\int e^{ikx}(f(x-y) - f(x))dx\right)\alpha y^{-1-\alpha}dy \\ &= \frac{-1}{\Gamma(1-\alpha)}\int (e^{iky} - 1)\hat{f}(k)\alpha y^{-1-\alpha}dy = (-ik)^\alpha\hat{f}(k),\end{aligned}$$

which generalizes the well known fact that $d^n f(x)/dx^n$ has Fourier transform $(-ik)^n\hat{f}(k)$ for derivatives of integer order. Clearly, the negative fractional derivative $d^\alpha f(x)/d(-x)^\alpha$ has Fourier transform $(ik)^\alpha\hat{f}(k)$. Then it follows from (18) using Theorem 2.2 that the generator L is of the form (16).

For $1 < \alpha < 2$ the proof is similar. Compute (see for example Lemma 7.3.8 in [13]) that

$$\int_0^\infty (e^{iky} - 1 - iky)\alpha y^{-\alpha-1}dy = \frac{\Gamma(2-\alpha)}{\alpha-1}(-ik)^\alpha$$

and note that since $1 < \alpha < 2$ the integral

$$b = \int_0^\infty \left(\frac{y}{1+y^2} - y\right)\alpha y^{-\alpha-1}dy$$

converges. Again we have (18) where all the constants are as before, and another application of Fubini shows that $d^\alpha f(x)/dx^\alpha$ has Fourier transform $(-ik)^\alpha \hat{f}(k)$ for all $f \in W^{2,1}(\mathbf{R^1})$. Then (16) follows easily.

If μ is stable with some index $0 < \alpha \leq 2$ then μ^t has a density $p(x,t)$. If $\alpha \neq 1,2$ then Theorem 2.3 applies, and this family of stable densities is the Green's function solution to the *fractional diffusion equation*

$$\frac{\partial C(x,t)}{\partial t} = -a_2 \frac{\partial C(x,t)}{\partial x} + Dq \frac{\partial^\alpha C(x,t)}{\partial (-x)^\alpha} + Dp \frac{\partial^\alpha C(x,t)}{\partial x^\alpha} \tag{19}$$

and the associated Lévy motion $\{A(t)\}$ is the stochastic solution to this fractional partial differential equation. If $\alpha = 2$ then (19) reduces to the classical diffusion equation, and the stochastic solution is Brownian motion with drift. The fractional diffusion equation is used in physics and hydrology to model *anomalous diffusion*, in which particles spread faster than the classical diffusion model predicts [3, 4, 5, 7, 8, 21, 23, 25].

Recently some multivariable analogues of (19) have been proposed [14, 15]. The multivariable fractional diffusion equation is

$$\frac{\partial C(x,t)}{\partial t} = -a \cdot \nabla C(x,t) + D \nabla^\alpha_M C(x,t) \tag{20}$$

where $\nabla^\alpha_M f(x)$ is the inverse Fourier transform of

$$\left[\int_{\|\theta\|=1} (-ik \cdot \theta)^\alpha M(d\theta) \right] \hat{f}(k) \tag{21}$$

and $M(d\theta)$ is an arbitrary probability measure on the unit sphere. Equation (20) is useful to model anomalous diffusion in more than one dimension. On $d = 1$ with $M\{+1\} = p$ and $M\{-1\} = q$ (20) reduces to (19). Stochastic solutions to (20) are multivariable Lévy motions, *i.e.*, Lévy processes whose distributions are multivariable stable. Multivariable stable laws are again defined by (12) but with Y, Y_i random vectors on $\mathbf{R^d}$. More generally, if we replace the norming constants a_n in (12) by linear operators A_n, we obtain the class of operator stable laws. Lévy processes whose distributions are operator stable are called operator Lévy motions. Operator Lévy motions are stochastic solutions to an abstract Cauchy problem $\partial C/\partial t = LC$ where L is the generator of the convolution semigroup associated with this operator Lévy motion. These generators are multivariable fractional derivative operators whose order of differentiation can vary with coordinate.

3. The fractional Cauchy problem

Zaslavsky [26] introduced the fractional kinetic equation

$$\frac{\partial^\beta g(x,t)}{\partial t^\beta} = Lg(x,t) + p_0(x) \frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{22}$$

for Hamiltonian chaos, where $0 < \beta < 1$, L is the generator (16) and $p_0 \in C^\infty(\mathbf{R^1})$ is an arbitrary initial condition. Here the fractional derivative $\partial^\beta g(x,t)/\partial t^\beta$ is the inverse Laplace transform of $s^\beta \tilde{g}(x,s)$, where $\tilde{g}(x,s) = \int_0^\infty e^{-st} g(x,t)\,dt$ is the usual Laplace transform. In the special case $p = 1/2$, when $A(t)$ is a symmetric stable law, Saichev and Zaslavsky [18] show that if $q(x,t) = T(t)p_0(x) = \int p_0(x-y)p(y,t)dy$ solves (19) then $g(x,t) = \int p_0(x-y)h(y,t)dy$ solves (22), where

$$h(x,t) = \frac{t}{\beta}\int_0^\infty p(x,\xi)g_\beta(\frac{t}{\xi^{1/\beta}})\xi^{-1/\beta-1}d\xi \tag{23}$$

and g_β is the density of a stable law with Laplace transform

$$\int_0^\infty e^{-st} g_\beta(t)\,dt = \exp(-s^\beta) \tag{24}$$

for all $s \geq 0$. This stable law is called the *stable subordinator*, see for example Samorodnitsky and Taqqu [19].

In order to motivate our main theorem, let us look at the result of Saichev and Zaslavsky from the point of view of convolution semigroups. A simple substitution in (23) yields

$$h(x,t) = \int_0^\infty p(x,(t/s)^\beta)g_\beta(s)ds \tag{25}$$

and so if we define a linear operator

$$S(t)f(x) = \int_0^\infty f(x-y)h(y,t)dy \tag{26}$$

then whenever $q(x,t) = T(t)f(x)$ solves (11), $g(x,t) = S(t)f(x)$ solves (22). Furthermore, it follows from (25) that

$$S(t)f = \int_0^\infty T((t/s)^\beta)g_\beta(s)f\,ds. \tag{27}$$

Now we come to the main result of this paper. We consider the general case where L is the generator of a Feller semigroup $\{T(t)\}$ associated with some infinitely divisible law on $\mathbf{R^d}$. We show that if $q(x,t) = T(t)f(x)$ solves (11) then $g(x,t) = S(t)f(x)$ solves the *fractional Cauchy problem* (22), where $S(t)$ is defined by (27). If μ^t has a density $p(x,t)$ for $t > 0$, so that $p(x,t)$ is the Green's function solution to (11), this means that the function $h(x,t)$ given by (23) is the Green's function solution to (22).

In the following, recall from [19] that the density g_β is a bounded continuous function with support in $\mathbf{R^1}_+$ and has Laplace transform (24). We define a *sectorial region* of the complex plane

$$\mathbf{C}(\alpha) = \{re^{i\theta} \in \mathbf{C} : r > 0, |\theta| < \alpha\}.$$

We call a family of linear operators on a Banach space X *strongly analytic in a sectorial region* if for some $\alpha > 0$ the mapping $t \mapsto T(t)f$ has an analytic extension to the sectorial region $\mathbf{C}(\alpha)$ for all $f \in X$ (see, e.g., [10] Sec. 3.12).

THEOREM 3.1. *Let $(X, \|\cdot\|)$ be a Banach space and L be the generator of a uniformly bounded, strongly continuous semigroup $\{T(t) : t \geq 0\}$. Then the family $\{S(t) : t \geq 0\}$ of linear operators from X into X given by (27), where g_β is given by (24), is uniformly bounded and strongly analytic in a sectorial region. Furthermore, $\{S(t) : t \geq 0\}$ is strongly continuous and $g(x,t) = S(t)p_0(x)$ solves*

$$\frac{\partial^\beta g(x,t)}{\partial t^\beta} = Lg(x,t) + p_0(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{28}$$

P r o o f. Since $\{T(t) : t \geq 0\}$ is uniformly bounded we have $\|T(t)f\| \leq M\|f\|$ for all $f \in X$. Bochner's Theorem ([1], Thm. 1.1.4 or [10], Thm. 3.7.4) implies that a function $F : \mathbf{R^1} \to X$ is integrable if and only if $F(s)$ is measurable and $\|F(s)\|$ is integrable, in which case

$$\left\|\int F(s)ds\right\| \leq \int \|F(s)\|ds.$$

Fix $f \in X$ and apply Bochner's Theorem with $F(s) = g_\beta(s)T((t/s)^\beta)f$ to see that

$$\begin{aligned}
\|S(t)f\| &= \left\|\int_0^\infty g_\beta(s)T((t/s)^\beta)f\,ds\right\| \\
&\leq \int_0^\infty \left\|g_\beta(s)T((t/s)^\beta)f\right\| ds \\
&= \int_0^\infty g_\beta(s)\left\|T((t/s)^\beta)f\right\| ds \\
&\leq \int_0^\infty g_\beta(s)M\|f\|ds = M\|f\|
\end{aligned}$$

since $g_\beta(s)$ is a probability density. This shows that $\{S(t) : t > 0\}$ is a well defined and uniformly bounded family of linear operators on X.

Given $f \in X$ and $\epsilon > 0$, choose s_0 such that $\int_0^{s_0} g_\beta(s)\,ds < \epsilon/4M\|f\|$. Since $\{T(t) : t \geq 0\}$ is strongly continuous and $T(0)f = f$ we have $\|T(t)f - f\| \to 0$ as $t \to 0$. Then we can also choose t_0 such that $\|T((t/s_0)^\beta)f - f\| < \epsilon/2$ for all $t \leq t_0$. Now another application of Bochner's Theorem shows that

$$\begin{aligned}
\|S(t)f - f\| &= \|\int_0^\infty g_\beta(s)(T((t/s)^\beta)f - f)\,ds\| \\
&\leq \int_0^{s_0} g_\beta(s)\|T((t/s)^\beta)f - f\|\,ds \\
&\quad + \int_{s_0}^\infty g_\beta(s)\|T((t/s)^\beta)f - f\|\,ds \\
&\leq 2M\|f\|\int_0^{s_0} g_\beta(s)\,ds + \sup_{s\geq s_0}\|T((t/s)^\beta)f - f\| < \epsilon
\end{aligned}$$

for all $t < t_0$. This shows that $\lim_{t\to 0} S(t)f = f$. Now if $t, h > 0$ then for all $f \in X$ we have $\|S(t+h)f - S(t)f\| \le \|S(t)\|\,\|S(h)f - f\| \le M\|S(h)f - f\| \to 0$ as $h \to 0$, and since the upper bound is independent of $t > 0$ we have that $\|S(t')f - S(t)f\| \to 0$ uniformly in $t > 0$ as $t' \to t$. This shows that $\{S(t) : t > 0\}$ is strongly continuous.

Now we want to show that

$$s^\beta \int_0^\infty e^{-s^\beta \xi} T(\xi) f\, d\xi = s \int_0^\infty e^{-st} S(t) f\, dt \tag{29}$$

for all $s > 0$ and all $f \in X$. In view of (24) we have

$$\begin{aligned} s^\beta \int_0^\infty & e^{-s^\beta \xi} T(\xi) f\, d\xi \\ &= \int_0^\infty \frac{-s}{\beta\xi} \frac{d}{ds} \left(e^{-s^\beta \xi} \right) T(\xi) f\, d\xi \\ &= \int_0^\infty \frac{-s}{\beta\xi} \frac{d}{ds} \left(\int_0^\infty e^{-s\xi^{1/\beta} t} g_\beta(t)\, dt \right) T(\xi) f\, d\xi \\ &= s \int_0^\infty \frac{1}{\beta\xi} \int_0^\infty e^{-s\xi^{1/\beta} t} \xi^{1/\beta} t g_\beta(t)\, dt\, T(\xi) f\, d\xi \end{aligned}$$

using dominated convergence to justify moving the derivative inside the integral. Substitute $t' = \xi^{1/\beta} t$ to obtain

$$\begin{aligned} & s \int_0^\infty \frac{1}{\beta\xi} \int_0^\infty e^{-st} \frac{t}{\xi^{1/\beta}} g_\beta\left(\frac{t}{\xi^{1/\beta}}\right) dt\, T(\xi) f\, d\xi \\ = \;& s \int_0^\infty e^{-st} \frac{t}{\beta} \int_0^\infty g_\beta\left(\frac{t}{\xi^{1/\beta}}\right) \xi^{-1/\beta - 1} T(\xi) f d\xi\, dt \end{aligned}$$

using the Fubini theorem (e.g.[10] Thm. 3.7.13 or [1] Thm. 1.1.9) and the fact that continuous functions are measurable ([1], Cor. 1.1.2). Finally substitute $\xi' = t/\xi^{1/\beta}$ to obtain

$$s \int_0^\infty e^{-st} \int_0^\infty g_\beta(\xi) T((t/\xi)^\beta) f\, d\xi\, dt = s \int_0^\infty e^{-st} S(t) f\, dt$$

proving (29) for all $s > 0$ and all $f \in X$.

Let $q(s) = \int_0^\infty e^{-st} T(t) f\, dt$ and $r(s) = \int_0^\infty e^{-st} S(t) f\, dt$ for any $s > 0$, so that we can write (29) in the form

$$s^\beta q(s^\beta) = s r(s) \tag{30}$$

for any $s > 0$. Now we want to show that this relation holds for certain complex numbers s. Fix $s \in \mathbf{C}_+ = \{z \in \mathbf{C} : \Re(z) > 0\}$, and let $F(t) = e^{-st} T(t) f$. Since F is continuous, it is measurable, and since $\|T(t) f\| \le M\|f\|$ we also have

$\|F(t)\| \le |e^{-st}|M\|f\| = e^{-t\Re(s)}M\|f\|$, so that the function $\|F(t)\|$ is integrable. Then Bochner's Theorem implies that $q(s) = \int F(t)dt$ exists for all $s \in \mathbf{C}_+$, with

$$\|q(s)\| = \left\|\int F(t)dt\right\| \le \int \|F(t)\|dt \le \int e^{-t\Re(s)}M\|f\|dt = \frac{M\|f\|}{\Re(s)}. \tag{31}$$

Since $q(s)$ is the Laplace transform of the bounded continuous function $t \mapsto T(t)f$, Theorem 1.5.1 of [1] shows that $q(s)$ is an analytic function on $s \in \mathbf{C}_+$.

Now we want to show that $r(s)$ is the Laplace transform of an analytic function defined on a sectorial region. Theorem 2.6.1 of [1] implies that if for some real x and some $\alpha \in (0, \pi/2]$ the function $r(s)$ has an analytic extension to the region $x + \mathbf{C}(\alpha + \pi/2)$ and if $\sup\{\|(s-x)r(s)\| : s \in x + \mathbf{C}(\alpha' + \pi/2)\} < \infty$ for all $0 < \alpha' < \alpha$, then there exists an analytic function $\bar{r}(t)$ on $t \in \mathbf{C}(\alpha)$ such that $r(s)$ is the Laplace transform of $\bar{r}(t)$. We will apply the theorem with $x = 0$. It follows from (30) that $r(s) = s^{\beta-1}q(s^\beta)$ for all $s > 0$, but the right hand side here is well defined and analytic on the set of complex s that are not on the branch cut and are such that $\Re(s^\beta) > 0$, so if $1/2 < \beta < 1$ then $r(s)$ has a unique analytic extension to the sectorial region $\mathbf{C}(\pi/(2\beta)) = \{s \in \mathbf{C} : \mathbf{Re}(\mathbf{s}^\beta) > \mathbf{0}\}$ (e.g., [10] Thm. 3.11.5), and note that $\pi/(2\beta) = \pi/2 + \alpha$ for some $\alpha > 0$. If $\beta < 1/2$ then $r(s)$ has an analytic extension to the sectorial region $s \in \mathbf{C}(\pi/2 + \alpha)$ for any $\alpha < \pi/2$, and $\Re(s^\beta) > 0$ for all such s. Now for any complex $s = re^{i\theta}$ such that $s \in \mathbf{C}(\pi/2 + \alpha')$ for any $0 < \alpha' < \alpha$, we have in view of (30) and (31) that

$$\begin{aligned} \|sr(s)\| &= \|s^\beta q(s^\beta)\| \\ &= \left|s^\beta\right| \|q(s^\beta)\| \\ &= \left|\frac{r^\beta e^{i\beta\theta}}{r^\beta \cos(\beta\theta)}\right| \|\Re(s^\beta)q(s^\beta)\| \\ &\le \frac{M\|f\|}{\cos(\beta\theta)} \end{aligned} \tag{32}$$

which is finite since $|\beta\theta| < \pi/2$. Hence Theorem 2.6.1 of [1] implies there exists an analytic function $\bar{r}(t)$ on $t \in \mathbf{C}(\alpha)$ with Laplace transform $r(s)$. Using the uniqueness of the Laplace transform (e.g., [1] Thm. 1.7.3), it follows that $t \mapsto S(t)f$ has an analytic extension (namely $t \mapsto \bar{r}(t)$) to the sectorial region $t \in \mathbf{C}(\alpha)$.

Next we wish to apply Theorem 2.6.1 of [1] again to show that for any $0 < \beta < 1$ the function

$$t \mapsto \int_0^t \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} S(u)f\,du \tag{33}$$

has an analytic extension to the same sectorial region $t \in \mathbf{C}(\alpha)$. A straightforward computation shows that

$$\int_0^\infty \frac{t^{-\beta}}{\Gamma(1-\beta)} e^{-st}dt = s^{\beta-1} \tag{34}$$

for any $\beta < 1$ and any $s > 0$. Since $r(s)$ is the Laplace transform of $t \mapsto S(t)f$, it follows from the convolution property of the Laplace transform (e.g., Property 1.6.4 [1]) that the function (33) has Laplace transform $s^{\beta-1}r(s)$ for all $s > 0$. Since $r(s)$ has an analytic extension to the sectorial region $s \in \mathbf{C}(\pi/2 + \alpha)$, so does $s^{\beta-1}r(s)$. For any $x > 0$, if $s = x + re^{i\theta}$ for some $r > 0$ and $|\theta| < \pi/2 + \alpha'$ for any $0 < \alpha' < \alpha$ then in view of (32) we have

$$\begin{aligned} \|(s-x)s^{\beta-1}r(s)\| &= \|(s-x)s^{\beta-2}sr(s)\| \\ &\leq r\|s^{\beta-2}\|\frac{M\|f\|}{\cos(\beta\theta)} \end{aligned}$$

where $\|s\|$ is bounded away from zero, $\|s\| \leq r + x$ and $\beta - 2 < -1$, so that $\|(s-x)s^{\beta-1}r(s)\|$ is bounded on the region $x + \mathbf{C}(\alpha' + \pi/2)\}$ for all $0 < \alpha' < \alpha$. Then it follows as before that the function (33) has an analytic extension to the sectorial region $t \in \mathbf{C}(\alpha)$.

Since $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup with generator L, Theorem 1.2.4 (b) in [16] implies that $\int_0^t T(s)f\,ds$ is in the domain of the operator L and

$$T(t)f = L\int_0^t T(s)f\,ds + f.$$

Since the Laplace transform $q(s)$ of $t \mapsto T(t)f$ exists, Corollary 1.6.5 of [1] shows that the Laplace transform of $t \mapsto \int_0^t T(s)f\,ds$ exists and equals $s^{-1}q(s)$. Corollary 1.2.5 [16] shows that L is closed. Fix s and let $g = q(s) = \int_0^\infty e^{-st}T(t)f\,dt$ and let g_n be a finite Riemann sum approximating this integral, so that $g_n \to g$ in X. Let $h_n = s^{-1}g_n$ and $h = s^{-1}g$. Then g_n, g are in the domain of L, $g_n \to g$ and $h_n \to h$. Since h_n is a finite sum we also have $L(h_n) = s^{-1}L(g_n) \to s^{-1}L(g)$. Since L is closed, this implies that h is in the domain of L and that $L(h) = s^{-1}L(g)$. In other words, the Laplace transform of $t \mapsto L\int_0^t T(s)f\,ds$ exists and equals $s^{-1}Lq(s)$. Then we have

$$\int_0^\infty e^{-st}T(t)f\,dt = s^{-1}L\int_0^\infty e^{-st}T(t)f\,dt + s^{-1}f$$

for all $s > 0$. Multiply through by s to obtain

$$s\int_0^\infty e^{-st}T(t)f\,dt = L\int_0^\infty e^{-st}T(t)f\,dt + f$$

and substitute s^β for s to get

$$s^\beta\int_0^\infty e^{-s^\beta t}T(t)f\,dt = L\int_0^\infty e^{-s^\beta t}T(t)f\,dt + f$$

for all $s > 0$. Now use (29) twice to get

$$s\int_0^\infty e^{-st}S(t)f\,dt = Ls^{1-\beta}\int_0^\infty e^{-st}S(t)f\,dt + f$$

and multiplying both sides by $s^{\beta-2}$ to obtain

$$s^{\beta-1}\int_0^\infty e^{-st}S(t)f\,dt = Ls^{-1}\int_0^\infty e^{-st}S(t)f\,dt + s^{\beta-2}f \tag{35}$$

where we have again used the fact that L is closed.

The term on the left hand side of (35) is $s^{\beta-1}r(s)$ which was already shown to be the Laplace transform of the function (33), which is analytic in a sectorial region. Equation (34) also shows that $s^{\beta-2}$ is the Laplace transform of $t \mapsto t^{1-\beta}/\Gamma(2-\beta)$. Now take the term $s^{\beta-2}f$ to the other side and invert the Laplace transforms. Using the fact that $\{S(t) : t \geq 0\}$ is uniformly bounded, we can apply the Phragmén-Mikusiński inversion formula for the Laplace transform (see [2], Cor.4) to obtain

$$\int_0^t \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)}S(u)f\,du - \frac{t^{1-\beta}}{\Gamma(2-\beta)}f = \lim_{n\to\infty} L\sum_{j=1}^{N_n}\alpha_{n,j}\frac{e^{c_{nj}t}}{c_{nj}}\int_0^\infty e^{-c_{nj}t}S(t)f\,dt,$$

where the constants N_n, $\alpha_{n,j}$, and c_n are given by the inversion formula and the limit is uniform on compact sets. Using again the fact that L is closed we get

$$\int_0^t \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)}S(u)f\,du - \frac{t^{1-\beta}}{\Gamma(2-\beta)}f = L\int_0^t S(u)f\,du \tag{36}$$

and since the function (33) is analytic in a sectorial region, the left side of (36) is differentiable for $t > 0$. Corollary 1.6.6 of [1] shows that

$$\frac{d}{dt}\int_0^t \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)}S(u)f\,du \tag{37}$$

has Laplace transform $s^\beta r(s)$ and hence (37) equals $d^\beta S(t)f/dt^\beta$. Now take the derivative with respect to t on both sides of (36) to obtain

$$\frac{d^\beta}{dt^\beta}S(t)f - \frac{t^{-\beta}}{\Gamma(1-\beta)}f = LS(t)f$$

for all $t > 0$, where we use the fact that L is closed to justify taking the derivative inside.

Our first application extends the result of Saichev and Zaslavsky [18] to include the case where the stable Lévy motion is not symmetric.

Corollary 3.2. *Suppose μ is stable with index $\alpha \neq 1, 2$ so that the family of densities $p(x,t)$ of μ^t are the Green's function solution to the fractional diffusion equation (19). Then the family of functions $h(x,t)$ given by (23) are the Green's function solution to (28) for any $0 < \beta < 1$, where L is the generator (16).*

4. Remarks

Operator Lévy motions are scaling limits of simple random walks, *c.f.* Example 1.2.19 in Meerschaert and Scheffler [13]. Operator Lévy motions provide a flexible model for anomalous diffusion in $\mathbf{R^d}$ when the rate of spreading varies with the coordinate, see Meerschaert, Benson and Bäumer [14]. These models assume that particle motions are uncorrelated in time. A tractable model with time correlation is the continuous time random walk (CTRW), a simple random walk subordinated to a renewal process. The random walk increments represent the magnitude of particle jumps, and the renewal epochs represent the times of the particle jumps. CTRW are used in physics to model a wide variety of phenomena connected with anomalous diffusion [11, 21, 25]. In many physical applications, the waiting time between renewals has infinite mean [22]. If this waiting time belongs to some β–stable domain of attraction, Meerschaert and Scheffler [12] show that the scaling limit is a subordinated operator Lévy motion. The subordinating process is the hitting time process for the β–stable subordinator, as described in Bertoin [6]. This subordinated operator Lévy motion is the stochastic solution to the fractional Cauchy problem (28) where L is the generator of the operator stable semigroup. In other words, long range correlation in time leads to subordination of the uncorrelated motion, which is modeled by a fractional time derivative.

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