Fractional diffusion with two time scales [1]

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Abstract

Moving particles that rest along their trajectory lead to time-fractional diffusion equations for the scaling limit distributions. For power law waiting times with infinite mean, the equation contains a fractional time derivative of order between 0 and 1. For finite mean waiting times, the most revealing approach is to employ two times scales, one for the mean and another for deviations from the mean. For finite mean power law waiting times, the resulting equation contains a first derivative as well as a derivative of order between 1 and 2. Finite variance waiting times lead to a second order partial differential equation in time. In this article we investigate the various solutions with regard to moment growth and scaling properties, and show that even infinite mean waiting times do not necessarily induce sub-diffusion, but can lead to super-diffusion if the jump distribution has nonzero mean.

Key words: Continuous time random walks, anomalous diffusion, fractional derivatives, power laws, hitting times.
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1 Introduction

Anomalous diffusion occurs when a cloud of particles spreads at a rate different from the classical square root of time law. Continuous time random walks (CTRW) can be used to derive governing equations for anomalous diffusion [1,2]. The CTRW model was introduced in [3,4]. Some recent surveys of CTRW models, their diverse applications in physics, and connections with fractional governing equations can be found in [5–8]. The CTRW is a stochastic process model for the movement of an individual particle consisting of a random waiting time between randomly sized particle jumps. The random walk can be separated into two distinct processes. One random process, $\mathbf{S}(n)$, describes the location of the particle in space after n jumps, the other, $N(t)$ characterises the number of jumps by time t. The particle location by time t is then described through subordination by $\mathbf{S}(N(t))$. The space-time vector process $(\mathbf{S}([t]), N(t))$ converges in a scaling limit to a simpler form $(\mathbf{A}(t), E(t))$, so that $\mathbf{S}(N(t))$ converges to $\mathbf{A}(E(t))$, and the density of the subordinated process $\mathbf{A}(E(t))$ solves a (fractional-order) governing equation for the motion of a particle undergoing anomalous diffusion [9–12].

In this article we investigate the limit process $E(t)$. This non-Markovian subordinator $E(t)$ is the inverse or first passage time process for a totally positively skewed γ-stable Lévy motion with positive drift. Then we apply these results to characterise the growth behaviour for CTRW scaling limits in the case where the particle jumps have finite second moments. We include the important case, often neglected in the CTRW literature, where the scaling limit $\mathbf{A}(t)$ of the particle jumps is a Brownian motion with drift. This limit derives from using two different scales in space, one for the mean jump and another for the deviation from the mean.

Using two scales may seem unnatural, but it is actually quite physical. Consider the case where the particle jumps have nonzero mean and finite variance. Scaling limits of this process can be understood in terms of examining the particle diffusion in the long time limit. As the time scale grows, the mean particle displacement grows at the same rate, but the displacement from that mean grows at a slower rate, proportional to the square root of the time scale. Using two spatial scales is necessary to preserve the detail, which ultimately leads to a Brownian motion with drift in the scaling limit. For a CTRW with finite mean waiting times, the same logic applies. Using two time scales preserves detail in the limit process that would otherwise be lost, and leads to a richer set of stochastic models for anomalous diffusion.

2 The model

In the usual CTRW formalism [1], the long-time scaling limit for the waiting time process is a γ-stable subordinator $W(s)$. Then the inverse Lévy process $E(t) = \inf\{s > 0 : W(s) > t\}$ counts the number of particle jumps by time $t \geq 0$, reflecting the fact that the time T_n of the nth particle jump and the number $N_t = \max\{n : T_n \leq t\}$ of jumps by time t are also inverse processes. When the waiting times between particle jumps have heavy tails with $0 < \gamma < 1$, subordination of the particle location process $\mathbf{A}(t)$ via the inverse Lévy process $E(t)$ is necessary in the long-time limit to account for the random waiting times, which leads to a time derivative of order γ in the governing equation [2]. In this case ($0 < \gamma < 1$) the random variable $s = E(t)$ has the probability density

$$p(t,s) := \frac{t}{\gamma s (as)^{1/\gamma}} g_\gamma\left(\frac{t}{(as)^{1/\gamma}}\right). \tag{1}$$

Here g_γ is the probability density of the standard γ-stable subordinator, so that its Laplace transform is

$$\int_0^\infty e^{-\lambda t} g_\gamma(t)dt = e^{-\lambda^\gamma}, \tag{2}$$

and a is the scaling parameter of the subordinator, so that

$$\int_0^\infty e^{-\lambda t}\frac{1}{a^{1/\gamma}} g_\gamma(t/a^{1/\gamma})dt = e^{-a\lambda^\gamma}.$$

When the probability distribution of the waiting times between particle jumps has heavy tails of order $1 < \gamma \leq 2$, meaning that the probability of waiting longer than t falls off like $t^{-\gamma}$, a different model is needed. In this case, convergence of the waiting time process is facilitated by centering to the mean waiting time w, which is not possible when $0 < \gamma < 1$. Accounting for this leads to a waiting time limit process $W(s)$ that is a completely positively skewed [2] stable Lévy process with index γ [10,13]. The inverse or first passage time process $E(t) = \inf\{s : W(s) \geq t\}$ counts the number of particle jumps by time t in the scaling limit. In [13] we showed that the distribution $H(s,t) = P(E(t) \leq s)$ of the first passage time process $E(t)$ with mean waiting time $w = 1$ has the following Laplace-Laplace transform:

$$\int_0^\infty \int_0^\infty e^{-us-\lambda T} H(s,T)\, ds\, dT = \frac{1 - a\lambda^{\gamma-1} + u/q_a(u)}{u(u+\lambda - a\lambda^\gamma)}, \tag{3}$$

where q_a is the unique analytic function that satisfies

$$aq_a(\lambda)^\gamma - q_a(\lambda) = \lambda$$

[2] The skew is irrelevant in the normal case $\gamma = 2$.

for all λ in a region containing the right half plane. Furthermore, we inverted (3) in [13], which, using scaling in t, (i.e.; replacing t by t/w) reads for general average waiting time w,

$$\Pr\{E(t) \leq s\} = \int_{\frac{t/w-s}{(as)^{1/\gamma}}}^{\infty} g_\gamma(u)\,du + \int_0^s \frac{m_a(s-u)}{(au)^{1/\gamma}} g_\gamma\left(\frac{t/w-u}{(au)^{1/\gamma}}\right) du, \tag{4}$$

where the function m_a satisfies $\int_0^\infty e^{-\lambda t} m_a(t)\,dt = 1/q_a(\lambda)$. The second term in (4) compensates for the fact that the waiting time process $W(s)$ is not monotone, and this term becomes negligible for t large [10, Remark 3.2].

Rescaling time as above, so that the average waiting time is one time unit, is always possible as long as we have a finite mean waiting time. However, the parameter a is then not the usual scaling parameter, but its time-normalised version; i.e., if the characteristic function of $W(s)$ is

$$E[e^{ikW(s)}] = e^{s(wik+\bar{a}(-ik)^\gamma)},$$

the mean normalised function $W_1(s) = w^{-1}W(s)$ has characteristic function

$$\begin{aligned} E[e^{ikW_1(s)}] &= e^{s(wik/w+\bar{a}(-ik/w)^\gamma)} \\ &= e^{s(ik+\bar{a}/w^\gamma(-ik)^\gamma)} = e^{s(ik+a(-ik)^\gamma)}. \end{aligned}$$

Hence

$$a = \bar{a}/w^\gamma. \tag{5}$$

Rewriting (4) in terms of $\bar{a}$ we obtain

$$\Pr\{E(t) \leq s\} = \int_{\frac{t-ws}{(\bar{a}s)^{1/\gamma}}}^{\infty} g_\gamma(u)\,du + \int_0^s \frac{wm_{\bar{a}/w^\gamma}(s-u)}{(\bar{a}u)^{1/\gamma}} g_\gamma\left(\frac{t-wu}{(\bar{a}u)^{1/\gamma}}\right) du. \tag{6}$$

In the following we choose to continue using the time normalised parameter a instead of $\bar{a}$, noting that with relation (5) we can easily switch from one to the other.

3 Dimensional Analysis

In order to effectively investigate and compare the first passage time densities, we bring them into dimensionless form. In (4) we have an expression with four variables, t, w, a, and s. The dimensions are $[T]$*time* for t, $[B]$*bulk jumps* (a counting unit for the number of jumps s, could also be taken dimensionless), the average waiting time w has dimension of $[T]/[B]$ (time per jump). In order for (4) to be dimensionally correct the argument of g_γ has to be dimensionless, and hence $\frac{t/w-s}{(as)^{1/\gamma}}$ has to be dimensionless. This forces the variable a to have a dimension of $[B^{\gamma-1}]$ (or $\bar{a}$ to have a dimension of $[B^{\gamma-1}]\left(\frac{[T]}{[B]}\right)^{\gamma} = [T]^{\gamma}/[B]$).

In order to non-dimensionalise (4) we need to obtain a scaling property for the function m_a. Now, $m_a(ct)$ has the following Laplace transform:

$$\begin{aligned}\int_0^\infty e^{-\lambda t} m_a(ct)\,dt =& \frac{1}{c}\int_0^\infty e^{-\lambda t/c} m_a(t)\,dt\\ =& \frac{1}{c}\frac{1}{q_a(\lambda/c)}.\end{aligned}$$

Since

$$aq_a(\lambda/c)^{\gamma} - q_a(\lambda/c) = \lambda/c, \tag{7}$$

we obtain that

$$\frac{a}{c^{\gamma-1}}\left(cq_a(\lambda/c)\right)^{\gamma} - cq_a(\lambda/c) = \lambda. \tag{8}$$

Substitute $a/c^{\gamma-1}$ for a in (7), compare with (8) and use the uniqueness of the function q_a in (7) established in [13, Lemma 3.1] to see that $cq_a(\lambda/c) = q_{a/c^{\gamma-1}}(\lambda)$ and thus

$$\begin{aligned}\int_0^\infty e^{-\lambda t} m_a(ct)\,dt =& \frac{1}{q_{a/c^{\gamma-1}}(\lambda)}\\ =& \int_0^\infty e^{-\lambda t} m_{a/c^{\gamma-1}}(t)\,dt.\end{aligned}$$

Therefore, in view of the uniqueness of the Laplace transform

$$m_a(ct) = m_{a/c^{\gamma-1}}(t).$$

Our goal is to find a parametrisation of (4) such that the parameters are dimensionless and the number of parameters is reduced. Introducing

$$\xi := w/(t/s) = ws/t \text{ and } \rho := a/(t/w)^{\gamma-1}$$

we obtain that

$$\begin{aligned}&\psi(t,s,a,w) = \Pr\{E(t) \le s\}\\ &= \int_{\frac{1-\xi}{(\rho\xi)^{1/\gamma}}}^\infty g_\gamma(u)\,du + \int_0^{\xi t/w} \frac{m_{\rho(t/w)^{\gamma-1}}(\xi t/w - u)}{(\rho(t/w)^{\gamma-1}u)^{1/\gamma}}\\ &\qquad\times g_\gamma\left(\frac{t/w-u}{(\rho(t/w)^{\gamma-1}u)^{1/\gamma}}\right)du\\ &= \int_{\frac{1-\xi}{(\rho\xi)^{1/\gamma}}}^\infty g_\gamma(u)\,du\\ &\quad + \int_0^{\xi} \frac{m_\rho(\xi-u)}{(\rho u)^{1/\gamma}} g_\gamma\left(\frac{1-u}{(\rho u)^{1/\gamma}}\right)du = \psi(\xi,\rho).\end{aligned}$$

The variables ρ and ξ are dimensionless versions of the mean waiting time w between particle jumps and the spread a of the deviation from the mean waiting time between jumps, respectively.

4 Plots of First Passage Time Densities

In this section we plot probability densities of the first passage time process $E(t)$ for the waiting times in a CTRW scaling limit. These are the hitting times of positively skewed Lévy motions (with drift if $\gamma > 1$).

4.1 The case $\gamma < 1$.

The cumulative distribution function for the first passage time of the stable subordinator is given by

$$\Pr\{E(t) \leq s\} = \int_{\frac{t}{(as)^{1/\gamma}}}^{\infty} g_\gamma(u)\, du$$

as can easily be deduced from (1). This formula can also be computed using Mittag-Leffler distributions [1,14,15]. Using $\xi = as/t^\gamma$ we obtain its density in dimensionless form

$$p(\xi) = \frac{1}{\gamma \xi^{1+1/\gamma}} g_\gamma \left(\frac{1}{\xi^{1/\gamma}} \right).$$

The plots of the densities are shown in Figure 1.

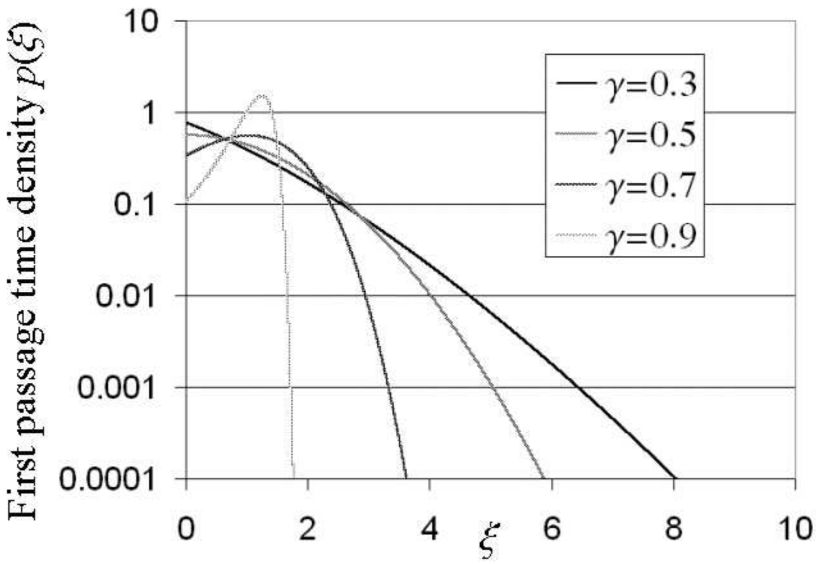

Fig. 1. First passage time densities of γ-stables, $\gamma < 1$.

4.2 *The case* $1 < \gamma < 2$.

Note that the dimensionless scale parameter ρ for the waiting time between particle jumps is proportional to $t^{1-\gamma}$. In Figure 2 we show the evolution of the arrival densities for selected values of γ and ρ. They were computed by changing the Laplace transform into a Fourier transform and then inverting using the Fast Fourier Transform algorithm.

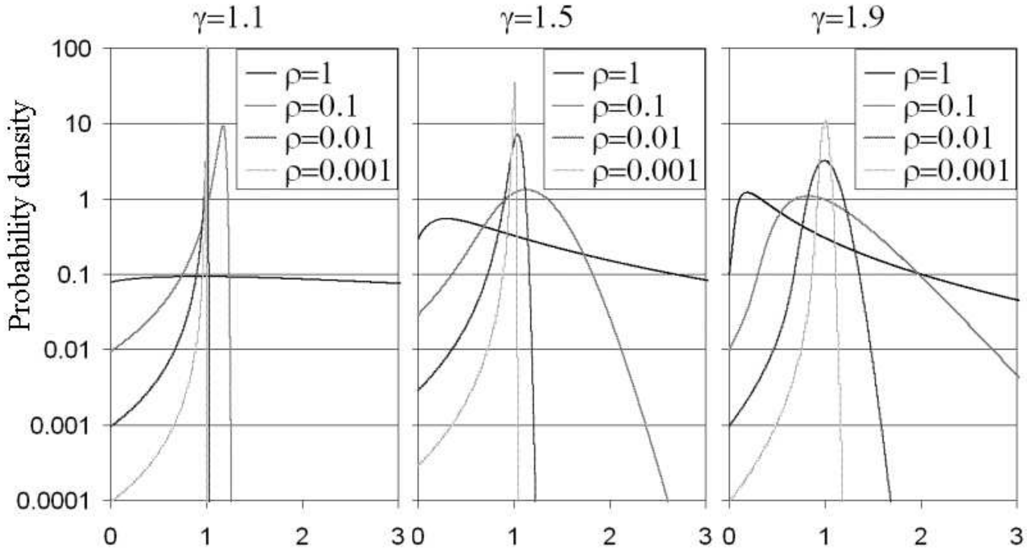

Fig. 2. First passage time densities, $1 < \gamma < 2$.

4.3 *The case* $\gamma = 2$

If the CTRW waiting times have finite second moments (the case $\gamma = 2$), then the limit procedure with two time scales produces a limit process $E(t)$ that is the hitting time distribution of a Brownian motion with drift. It is well-known that this hitting time distribution is given by the the inverse Gaussian [16].

Since the hitting time process $E(t)$ is not Markovian, the scaling limit $\mathbf{A}(E(t))$ of this continuous time random walk is not a Markov process. One simple Markovian model of particle motion would replace the subordinator $E(t)$ with a Poisson process. In order to compare these two models, in Figure 3 we plotted the scale-adjusted Poisson against the hitting time density, the inverse Gaussian, for various ρ. For large ρ (early time) there is an obvious difference: The inverse Gaussian hitting times are shorter, allowing for more frequent jumps, as the probability of several jumps in a small amount of time is larger than in the Markovian case. Hence this CTRW limit process is significantly non-Markovian.

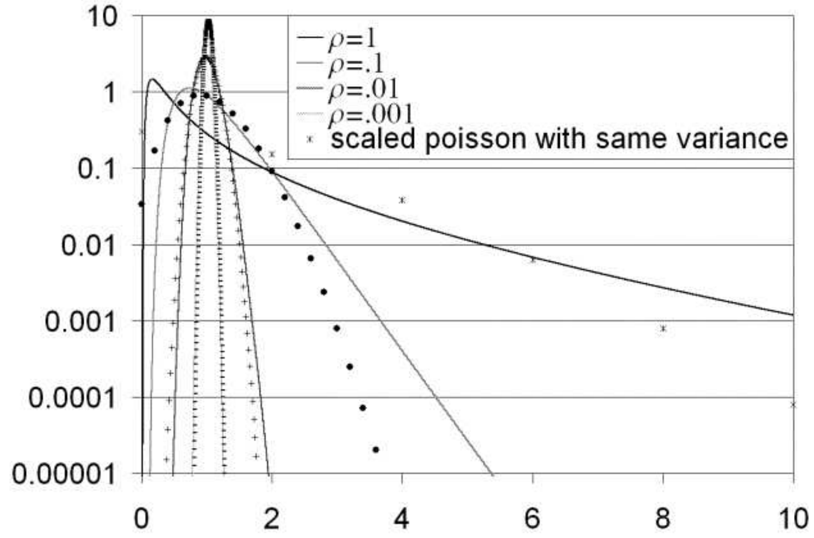

Fig. 3. Poisson vs. Inverse Gaussian

5 The first and second moments

In this section we compute the first and second moments of the waiting time process $E(t)$ that is used to subordinate the process of particle jumps $\mathbf{A}(t)$ to model particle location at time t in the scaling limit. We show that for $\gamma < 1$ the first moment grows like t^γ, while the variance grows like $t^{2\gamma}$. For $\gamma > 1$ we show that for large time the first moment grows like t while the variance grows like $t^{3-\gamma}$.

5.1 The case $\gamma < 1$

In order to compute the moments of $E(t)$ in the case of $\gamma < 1$ we compute the Laplace-Laplace transform of its density $p(t,s) = \frac{t}{\gamma s (as)^{1/\gamma}} g_\gamma(t/(as)^{1/\gamma})$. Using the dominated convergence theorem and the definition of g_γ given by (2), we see that for $s > 0$

$$\begin{aligned}\tilde{p}(\lambda, s) :=& \int_0^\infty e^{-\lambda t} p(t,s)\, dt = -\frac{1}{\gamma s} \int_0^\infty \left(\frac{d}{d\lambda} e^{-\lambda t} \right) \frac{1}{(as)^{1/\gamma}} g_\gamma \left(\frac{t}{(as)^{1/\gamma}} \right) dt \\ =& -\frac{1}{\gamma s} \frac{d}{d\lambda} \int_0^\infty e^{-\lambda (as)^{1/\gamma} u} g_\gamma(u)\, du = -\frac{1}{\gamma s} \frac{d}{d\lambda} e^{-(\lambda (as)^{1/\gamma})^\gamma} \\ =& a \lambda^{\gamma - 1} e^{-as\lambda^\gamma}.\end{aligned}$$

Taking the Laplace transform in s yields

$$\begin{aligned}\tilde{\tilde{p}}(\lambda, u) &= \int_0^\infty e^{-us}\tilde{p}(\lambda, s)\,ds = \int_0^\infty e^{-us} a\lambda^{\gamma-1} e^{-as\lambda^\gamma}\,ds \\ &= a\lambda^{\gamma-1}\int_0^\infty e^{-s(u+a\lambda^\gamma)}\,ds = \frac{a\lambda^{\gamma-1}}{u+a\lambda^\gamma}.\end{aligned}$$

In order to compute the first moment $\mu(t) = \int_0^\infty sp(t,s)\,ds$, note that

$$\mu(t) = -\frac{d}{du}\left(\int_0^\infty e^{-us}p(t,s)\,ds\right)\bigg|_{u=0}.$$

Hence,

$$\tilde{\mu}(\lambda) = \int_0^\infty e^{-\lambda t}\mu(t)\,dt = -\frac{d}{du}\left(\frac{a\lambda^{\gamma-1}}{u+a\lambda^\gamma}\right)\bigg|_{u=0} = \frac{\lambda^{-1-\gamma}}{a}.$$

Thus, for $\gamma < 1$, we obtain using the well-known Laplace transform formula $\int_0^\infty e^{-st}t^{-\beta}dt = \Gamma(1-\beta)s^{\beta-1}$ that

$$\mu(t) = \frac{t^\gamma}{a\Gamma(1+\gamma)}. \tag{9}$$

Similarly, the p^{th} moment is computed by

$$\tilde{\mu}_p(\lambda) = (-1)^p\frac{d^p}{du^p}\left(\frac{a\lambda^{\gamma-1}}{u+a\lambda^\gamma}\right)\bigg|_{u=0} = \frac{p!\lambda^{-1-p\gamma}}{a^p},$$

leading to

$$\mu_p(t) = \frac{p!}{a^p\Gamma(1+p\gamma)}\,t^{p\gamma}.$$

This extends the result in [1, Corollary 3.2] by giving the exact form of the constant in front of $t^{p\gamma}$.

We conclude by computing the variance,

$$\sigma^2(t) = \frac{t^{2\gamma}}{a^2}\left(\frac{2}{\Gamma(1+2\gamma)} - \frac{1}{\Gamma(1+\gamma)^2}\right).$$

5.2 The case $1 < \gamma < 2$

We use the same technique to compute the first and second moments of the first arrival processes for $\gamma > 1$. Recall that the Laplace-Laplace transform of the mean normalised first passage time density is given by (3). Thus

$$
\begin{aligned}
\int_0^\infty e^{-\lambda t}\mu_1(t)dt &:= \int_0^\infty e^{-\lambda t}\int_0^\infty s\,dH(s,t)dt \\
&= -\frac{d}{du}\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-us}\,dH(s,t)dt\Big|_{u=0} \\
&= -\frac{d}{du}\frac{1-a\lambda^{\gamma-1}+u/q_a(u)}{u+\lambda-a\lambda^\gamma}\Big|_{u=0} \\
&=\frac{1-a\lambda^{\gamma-1}}{(\lambda-a\lambda^\gamma)^2}-\frac{1/q_a(0)}{\lambda-a\lambda^\gamma} \\
&=\frac{1-a^{1/(\gamma-1)}\lambda}{\lambda^2(1-a\lambda^{\gamma-1})}=\frac{1-a^{1/(\gamma-1)}\lambda}{a\lambda^{\gamma+1}(\frac{1}{a\lambda^{\gamma-1}}-1)} \\
&=-\,\frac{1-a^{1/(\gamma-1)}\lambda}{a\lambda^{\gamma+1}}\sum_{n=0}^\infty(1/a\lambda^{\gamma-1})^n
\end{aligned}
\tag{10}
$$

This inverts to

$$
\begin{aligned}
\mu_1(t) = \sum_{n=0}^\infty (t^{\gamma-1}/a)^n \Bigg(&-\frac{t^\gamma}{a\Gamma(1+\gamma+n(\gamma-1))} \\
&+\frac{t^{\gamma-1}}{a^{\frac{2-\gamma}{1-\gamma}}\Gamma(\gamma+n(\gamma-1))}\Bigg).
\end{aligned}
$$

Note that $q_a(0) = a^{1/(1-\gamma)}$. Unfortunately, this series does not reveal the type of growth of $\mu(t)$ for large t. The late time growth behaviour is revealed by the following asymptotic expansion, obtained by expanding equation (10) around $\lambda = 0$; i.e.

$$
\tilde{\mu}_1(\lambda) = \frac{1-a^{1/(\gamma-1)}\lambda}{\lambda^2}\sum_{n=0}^\infty (a\lambda^{\gamma-1})^n
$$

Theorem 1 *Let $1 < \gamma \leq 2$. Then the first moment of the hitting time density at time $t > 0$ for a stable Lévy motion with drift with mean velocity one and shape factor a satisfies*

$$
\mu_1(t) = \sum_{n=0}^N \left[\frac{t(at^{1-\gamma})^n}{\Gamma(2+n(1-\gamma))} - a^{1/(\gamma-1)}\frac{(at^{1-\gamma})^n}{\Gamma(1+n(1-\gamma))}\right] + o(t^{1-N(\gamma-1)}) \tag{11}
$$

for all integers $N \geq 0$ and $t \to \infty$. Terms are set equal to zero if they contain $\Gamma(k)$ for some negative integer k.

The second moment satisfies

$$\begin{aligned}\mu_2(t) = \sum_{n=0}^{N} \Bigg[& \frac{2t^2(n+1)}{\Gamma(3+n(1-\gamma))}(at^{1-\gamma})^n - \frac{2a^{1/(\gamma-1)}t(n+1)}{\Gamma(2+n(1-\gamma))}(at^{1-\gamma})^n \\ & - \frac{2a^{2/(\gamma-1)}}{(\gamma-1)\Gamma(1+n(1-\gamma))}(at^{1-\gamma})^n \Bigg] + o(t^{2-N(\gamma-1)})\end{aligned} \tag{12}$$

for all integers $N \geq 0$ and $t \to \infty$. Again, terms are taken to be zero if they contain $\Gamma(k)$ for some negative integer k.

This gives us an estimate for the variance for large t (assuming that the mean waiting time is one):

$$\begin{aligned}\sigma^2(t) =& 2t^2 \sum_{n=0}^{1} \frac{n+1}{\Gamma(3+n(1-\gamma))}(at^{1-\gamma})^n + o(t^{3-\gamma}) \\ & - \left(t \sum_{n=0}^{1} \frac{(at^{1-\gamma})^n}{\Gamma(2+n(1-\gamma))} + o(t^{2-\gamma}) \right)^2 \\ =& \left(\frac{4a}{\Gamma(4-\gamma)} - \frac{2a}{\Gamma(3-\gamma)} \right) t^{3-\gamma} + o(t^{3-\gamma}).\end{aligned}$$

In case where the mean waiting time is not one we can use the scaling procedure outlined in Section 2: Substituting $\bar{a}/w^\gamma$ for a and t/w for t yields

$$\sigma^2(t) = \left(\frac{4\bar{a}}{\Gamma(4-\gamma)} - \frac{2\bar{a}}{\Gamma(3-\gamma)} \right) \frac{t^{3-\gamma}}{w^3} + o(t^{3-\gamma}).$$

5.3 The case $\gamma = 2$

Here the previous calculations are much simplified. Recall that $q_a(0)$ satisfies $aq_a(0)^\gamma - q_a(0) = 0$, or in this case, $q_a(0) = 1/a$. Using equation (10) we obtain that

$$\tilde{\mu}(\lambda) = \frac{1-a\lambda}{\lambda^2(1-a\lambda)} = 1/\lambda^2.$$

For the second moment we obtain

$$\begin{aligned}\tilde{\mu}_2(\lambda) =& \frac{2(1-a\lambda-a^2\lambda^2(1-a\lambda))}{\lambda^3(1-a\lambda)^2} \\ =& \frac{2(1+a\lambda)}{\lambda^3} = 2/\lambda^3 + 2a/\lambda^2\end{aligned}$$

This yields that $\mu(t) = t$ and

$$\sigma^2(t) = t^2 + 2at - t^2 = 2at. \tag{13}$$

If the mean waiting time is not one, we obtain

$$\sigma^2(t) = 2\bar{a}t/w^3.$$

6 Moment growth for CTRW scaling limits

In this section we show how the formulas for the moments of the subordinator $E(t)$, obtained in the previous section, can be used to compute moments of the process $\mathbf{A}(E(t))$ that represents CTRW particle location at time t in the long-time scaling limit. If the particle jumps have zero mean and finite second moments, then $\mathbf{A}(t)$ is a Brownian motion with no drift, i.e., the random vector $\mathbf{A}(t)$ has mean zero and the variance of any component grows linearly with time. On the other hand, if the jumps have non-zero mean, then a two-scale limiting procedure in which the mean jump and the deviation from the mean are treated separately leads to a Brownian motion with drift in the scaling limit [18, Exercise 10.8]. Then both the mean and variance of each component of the vector process $\mathbf{A}(t)$ grow linearly with time. In either case, if $\mathbf{A}(t)$ has vector mean $\mathbf{m}(t)$ and covariance matrix $C(t)$, then a simple conditioning argument shows that the subordinated process $\mathbf{A}(E(t))$ has a mean $\mathbf{m}_S(t)$ whose ith coordinate is

$$\mathbf{m}_S(t)_i = \int_0^\infty \mathbf{m}(s)_i \; d_s H(t,s) \tag{14}$$

and covariance matrix $C_S(t)$ whose ij entry is given by

$$C_S(t)_{i,j} = \int_0^\infty \Big(C_{i,j}(s) + \mathbf{m}(s)_i \mathbf{m}(s)_j \Big) \, d_s H(t,s) - \mathbf{m}_S(t)_i \mathbf{m}_S(t)_j.$$

6.1 Subordinated Brownian motion

Assume that the particle jump process $\mathbf{A}(t)$ is a vector Brownian motion with zero mean, so that $\mathbf{m}(t) = \mathbf{m}_S(t) = 0$, and covariance matrix $C(t) = tD$ for some positive definite matrix D. For this process, the classical model for particle diffusion, the variance of each component grows linearly with time.

In the case of $\gamma < 1$, by virtue of (9), the covariance matrix

$$C_S(t) = \frac{t^\gamma}{a\Gamma(1+\gamma)} D$$

and hence the subordinated process is subdiffusive, meaning that the variance of any component grows slower than in the original diffusion process.

In the case of $1 < \gamma \leq 2$, using (11), we obtain that the covariance matrix

$$C_S(t) = \left(t + \frac{at^{2-\gamma}}{\Gamma(3-\gamma)} - a^{1/(\gamma-1)} + o(t^{2-\gamma})\right) D \tag{15}$$

as $t \to \infty$. This process is called "diffusive" since the variance of any component also grows linearly with time. In practice this may be mistaken for subdiffusive (slower than linear) spreading, since the linear term does not dominate until late time. In the case $\gamma = 2$, using that $\mu(t) = t$ we obtain the somewhat surprising result that the time scaling parameter a has no bearing on the resulting plume spreading, as $C_S(t) = tD$.

6.2 Subordinated Brownian motion with drift

Assume that the particle jump process $\mathbf{A}(t)$ is a vector Brownian motion with drift, so that the mean $\mathbf{m}(t) = \mathbf{v}t$, and the covariance matrix $C(t) = tD$ for some positive definite matrix D. This model is also diffusive since the variance grows linearly.

In case $\gamma < 1$, the delay affects the mean plume location. Substituting (9) into (14) we obtain

$$\mathbf{m}_S(t) = \frac{t^\gamma}{a\Gamma(1+\gamma)}\mathbf{v}.$$

The effect on the covariance is surprising as for $\gamma > 0.5$ the inclusion of the drift gives rise to a super-diffusive process:

$$C_S(t)_{i,j} = \frac{t^\gamma}{a\Gamma(1+\gamma)} D_{ij} + t^{2\gamma}\left(\frac{2}{a^2\Gamma(1+2\gamma)} - \frac{1}{(a\Gamma(1+\gamma))^2}\right)\mathbf{v}_i\mathbf{v}_j.$$

In case $1 < \gamma < 2$, we obtain

$$\mathbf{m}_S(t) = \left(t + \frac{at^{2-\gamma}}{\Gamma(3-\gamma)} + o(t^{2-\gamma})\right)\mathbf{v}.$$

For the covariance we compute

$$\begin{aligned} C_S(t)_{i,j} = &\left(t + \frac{at^{2-\gamma}}{\Gamma(3-\gamma)}\right) D_{ij} \\ &+ \left(at^{3-\gamma}\left(\frac{4}{\Gamma(4-\gamma)} - \frac{2}{\Gamma(3-\gamma)}\right) + o(t^{3-\gamma})\right)\mathbf{v}_i\mathbf{v}_j \end{aligned}$$

Note that the highest order terms in the respective covariances are proportional to the square of the velocity, confirming the argument that the high rate of dispersion is due to a smearing out effect rather than an inherent super-diffusion.

For $\gamma = 2$, we see, using (13), that the covariance grows linearly,

$$C_{i,j}(t) = tD_{ij} + 2at\mathbf{v}_i\mathbf{v}_j + t^2\mathbf{v}_i\mathbf{v}_j - t^2\mathbf{v}_i\mathbf{v}_j = (D_{ij} + 2a\mathbf{v}_i\mathbf{v}_j)t.$$

A Proof of Theorem 1

The proof requires the following simple lemma.

Lemma 2 *Let $0 < \alpha < \pi/2$, $n \geq 1$, and let $\lambda \in \mathbb{C}$ with $-\pi/2 - \alpha < \arg(\lambda) < \pi/2 + \alpha$. Then*

$$\left(\frac{d}{d\lambda}\right)^n \frac{1}{1 - a\lambda^{\gamma-1}} = \sum_{i=1}^{n} b_{i,n}\lambda^{i(\gamma-2)-(n-i)}(1 - a\lambda^{\gamma-1})^{-i-1}$$

for some constants $b_{i,n}$. For $|\lambda| \to \infty$ this implies that

$$\left(\frac{d}{d\lambda}\right)^n \frac{1}{1 - a\lambda^{\gamma-1}} = O(|\lambda|^{1-\gamma-n})$$

for all $n \geq 0$. For $\lambda \to 0$ we have

$$\left(\frac{d}{d\lambda}\right)^n \frac{1}{1 - a\lambda^{\gamma-1}} = \begin{cases} O(\lambda^{\gamma-1-n}) & \text{if } n \geq 1 \\ O(1) & \text{if } n = 0. \end{cases}$$

The case $n = 0$ is obvious and the remaining cases will be proven by induction. Clearly the lemma holds for $n = 1$. Assume it holds for some $n \geq 1$. Then

$$\begin{aligned}
&\frac{d}{d\lambda}\sum_{i=1}^{n} b_{i,n}\lambda^{i(\gamma-2)-(n-i)}(1-a\lambda^{\gamma-1})^{-i-1} \\
&= \sum_{i=1}^{n} b_{i,n}\Big((i(\gamma-2)-(n-i))\lambda^{i(\gamma-2)-(n-i)-1}(1-a\lambda^{\gamma-1})^{-i-1} + \\
&\qquad \lambda^{i(\gamma-2)-(n-i)}(-i-1)(1-a\lambda^{\gamma-1})^{-i-2}(-a)(\gamma-1)\lambda^{\gamma-2}\Big) \\
&= \sum_{i=1}^{n} b_{i,n}\Big((i(\gamma-2)-(n-i))\lambda^{i(\gamma-2)-(n+1-i)}(1-a\lambda^{\gamma-1})^{-i-1}\Big) + \\
&\qquad \sum_{j=2}^{n+1} b_{j-1,n}\Big(\lambda^{j(\gamma-2)-(n+1-j)}(-j)(1-a\lambda^{\gamma-1})^{-j-1}(-a)(\gamma-1)\Big) \\
&= \sum_{i=1}^{n+1} b_{i,n+1}\lambda^{i(\gamma-2)-(n+1-i)}(1-a\lambda^{\gamma-1})^{-i-1}
\end{aligned}$$

for $b_{i,n+1} = b_{i,n}(i(\gamma-2)-(n-i)) + b_{i-1,n}aj(\gamma-1)$ and with $b_{n+1,n} = b_{0,n} = 0$. Hence the formula is valid for $n+1$ and thus by induction the formula holds for all integers $n \geq 1$. $\square$

The proof of Theorem 1 is based on two results from Laplace transform theory. The first is the trivial observation that if the function f has Laplace transform $\tilde{f}$, then the Laplace transform of $t \mapsto t^n f(t)$ has Laplace transform

$$(-1)^n (d/d\lambda)^n \int_0^\infty e^{-\lambda t} f(t)\, dt.$$

The second is a Tauberian theorem for Laplace transforms (see, for example, [19] Thm. 2.6.4), which tells us that the behaviour of $\lambda\tilde{f}(\lambda)$ at zero corresponds to the behaviour of its inverse Laplace transform at infinity, as long as $\lambda\tilde{f}(\lambda)$ is bounded and analytic in a sectorial region containing the right half plane.

In order to prove (11), note that for all integers $M, N \geq 0$, equation (10) is equal to

$$\begin{aligned}
\hat{\mu}_1(\lambda) &= \frac{1-a^{1/(\gamma-1)}\lambda}{\lambda^2(1-a\lambda^{\gamma-1})} \\
&= \frac{1}{\lambda^2}\frac{(a\lambda^{\gamma-1})^{N+1} - a^{1/(\gamma-1)}\lambda(a\lambda^{\gamma-1})^{M+1}}{1-a\lambda^{\gamma-1}} \\
&\quad + \frac{1}{\lambda^2}\left(\sum_{n=0}^{N}(a\lambda^{\gamma-1})^n - a^{1/(\gamma-1)}\lambda\sum_{n=0}^{M}(a\lambda^{\gamma-1})^n\right) \\
&=: r1(\lambda) + r2(\lambda).
\end{aligned}$$

Using l'Hopital's rule, we see that $\hat{\mu}_1$ is continuously extendable across $\lambda = a^{-\frac{1}{\gamma-1}}$ with

$$\begin{aligned}\hat{\mu}_1(a^{-\frac{1}{\gamma-1}}) :=& \lim_{\lambda\to a^{-\frac{1}{\gamma-1}}} \frac{-a^{1/(\gamma-1)}}{2\lambda(1-a\lambda^{\gamma-1}) - a(\gamma-1)\lambda^{\gamma}} \\ =& \frac{a^{1/(\gamma-1)}}{a(\gamma-1)a^{-\frac{\gamma}{\gamma-1}}} = \frac{a^{2/(\gamma-1)}}{\gamma-1}.\end{aligned}$$

By the Riemann continuation theorem (see, for example, [20], Theorem 7.3.4), the extension is analytic and hence the extension of $\hat{\mu}_1$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$. Let $R \geq 0$. Pick the integers M, N such that

$$(M+1)(\gamma-1) > R \geq M(\gamma-1)$$

and

$$(N+1)(\gamma-1) > R+1 \geq N(\gamma-1).$$

Then $(d/d\lambda)^R r2(\lambda)$ inverts to

$$(-1)^R t^R \left(\sum_{n=0}^{N} \frac{t(at^{1-\gamma})^n}{\Gamma(2+n(1-\gamma))} - a^{1/(\gamma-1)} \sum_{n=0}^{M} \frac{(at^{1-\gamma})^n}{\Gamma(1+n(1-\gamma))} \right)$$

where terms that are not defined are taken to be zero (The respective derivatives of these terms on the Laplace transform side vanish). Now, the extension of $\hat{\mu}_1$ and $r2$ are analytic in a sectorial region containing the right halfplane, and thus the extension of $r1$ is analytic there as well and so is the extension of $(d/d\lambda)^R r1(\lambda)$ (derivatives of analytic functions are analytic functions).

We would like to show that $\lambda(d/d\lambda)^R r1(\lambda)$ is bounded in a sectorial region containing the right halfplane and converges to zero a λ goes to zero. Realise that

$$\begin{aligned}\left(\frac{d}{d\lambda}\right)^R r1(\lambda) = \sum_{i=0}^{R} \Big(& c_i \lambda^{(\gamma-1)(N+1)-2-i} \\ & - d_i \lambda^{(\gamma-1)(M+1)-1-i} \Big) \left(\frac{d}{d\lambda}\right)^{R-i} \frac{1}{1-a\lambda^{\gamma-1}}\end{aligned} \quad \text{(A.1)}$$

for some constants c_i, d_i. Using Lemma 2 we obtain that for $|\lambda| \to \infty$,

$$\begin{aligned}\lambda \left(\frac{d}{d\lambda}\right)^R r1(\lambda) =& O(\lambda^{1+(\gamma-1)(N+1)-2+1-\gamma-R}) + O(\lambda^{1+(\gamma-1)(M+1)-1+1-\gamma-R}) \\ =& O(\lambda^{N(\gamma-1)-(R+1)}) + O(\lambda^{M(\gamma-1)-R}).\end{aligned}$$

For $\lambda \to 0$ we write $r1(\lambda) = r10(\lambda) + r11(\lambda)$ where $r10(\lambda)$ contains the terms with $i < R$ and $r11(\lambda)$ contains the terms with $i = R$. Then we obtain that

$$\begin{aligned}\lambda \left(\frac{d}{d\lambda}\right)^R r10(\lambda) =& O(\lambda^{1+(\gamma-1)(N+1)-2+\gamma-1-R}) + O(\lambda^{1+(\gamma-1)(M+1)-1+\gamma-1-R}) \\ =& O(\lambda^{(N+1)(\gamma-1)-(R+2-\gamma)}) + O(\lambda^{(M+1)(\gamma-1)-(R+1-\gamma)})\end{aligned}$$

while

$$\lambda \left(\frac{d}{d\lambda}\right)^R r11(\lambda) = O(\lambda^{(N+1)(\gamma-1)-(R+1)}) + O(\lambda^{(M+1)(\gamma-1)-R}).$$

so that $r11(\lambda)$ dominates. Then it follows that the extension of $\lambda \left(\frac{d}{d\lambda}\right)^R r1(\lambda)$ is bounded in a sectorial region containing the right halfplane and converges to zero as λ goes to zero. By the Tauberian Theorem for Laplace transforms (see, for example, [19] Thm. 2.6.4) there exists a continuous function g such that $\int_0^\infty e^{-\lambda t} g(t)\, dt = \left(\frac{d}{d\lambda}\right)^R r1(\lambda)$ with $\lim_{t\to\infty} g(t) = 0$. Hence

$$\begin{aligned}(-t)^R \mu_1(t) = (-t)^R \Bigg(& \sum_{n=0}^{N} \frac{t(at^{1-\gamma})^n}{\Gamma(2+n(1-\gamma))} \\ & - a^{1/(\gamma-1)} \sum_{n=0}^{M} \frac{(at^{1-\gamma})^n}{\Gamma(1+n(1-\gamma))} \Bigg) + g(t)\end{aligned}$$

or

$$\mu_1(t) = \sum_{n=0}^{N} \frac{t(at^{1-\gamma})^n}{\Gamma(2+n(1-\gamma))} - a^{1/(\gamma-1)} \frac{(at^{1-\gamma})^n}{\Gamma(1+n(1-\gamma))} + o(t^{1-N(\gamma-1)})$$

as $t \to \infty$. The extra terms in the second sum can be added due to the "little o" term. Note that if (11) holds for one N_0, it holds for all $N < N_0$ as well. Hence we showed (11) for all $N \le (R+1)/(\gamma-1)$. Since R was chosen arbitrarily, we showed it for all $N \in \mathbb{N}$.

The proof for the second moment is analogous, using the fact that q_a is an inverse function, namely $aq_a(\lambda)^\gamma - q_a(\lambda) = \lambda$, which implies that $\frac{d}{d\lambda} q_a(\lambda) =$

$1/(a\gamma q_a(\lambda)^{\gamma-1}-1)$. Thus, we obtain for the second derivative that

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda t}\int_0^\infty s^2\,dH(s,t)dt = \frac{d^2}{du^2}\int_0^\infty e^{-\lambda t}\int_0^\infty e^{-us}\,dH(s,t)dt\bigg|_{u=0}\\
&=\left(\frac{d^2}{du^2}\frac{1-a\lambda^{\gamma-1}+u/q_a(u)}{u+\lambda-a\lambda^\gamma}\right)\bigg|_{u=0}\\
&=\frac{d}{du}\left(-\frac{1-a\lambda^{\gamma-1}+u/q_a(u)}{(u+\lambda-a\lambda^\gamma)^2}+\frac{1/q_a(u)-u/q_a(u)^2\frac{1}{a\gamma q_a(u)^{\gamma-1}-1}}{u+\lambda-a\lambda^\gamma}\right)\Bigg|_{u=0}\\
&=2\frac{1-a\lambda^{\gamma-1}}{(\lambda-a\lambda^\gamma)^3}-\frac{1/q_a(0)}{(\lambda-a\lambda^\gamma)^2}-\frac{1/q_a(0)}{(\lambda-a\lambda^\gamma)^2}\\
&\quad+\frac{-1/q_a(0)^2\frac{1}{a\gamma q_a(0)^{\gamma-1}-1}-1/q_a(0)^2\frac{1}{a\gamma q_a(0)^{\gamma-1}-1}}{\lambda-a\lambda^\gamma}\\
&=2\frac{1}{\lambda^3(1-a\lambda^{\gamma-1})^2}-2\frac{\lambda/q_a(0)}{\lambda^3(1-a\lambda^{\gamma-1})^2}-2\frac{1/q_a(0)\frac{1}{\gamma(aq_a(0)^\gamma-q_a(0))+(\gamma-1)q_a(0)}}{\lambda-a\lambda^\gamma}\\
&=2\frac{1}{\lambda^3(1-a\lambda^{\gamma-1})^2}-2\frac{\lambda/q_a(0)}{\lambda^3(1-a\lambda^{\gamma-1})^2}-2\frac{1/q_a(0)\frac{1}{(\gamma-1)q_a(0)}}{\lambda-a\lambda^\gamma}\\
&=2\frac{1-\lambda/q_a(0)-\lambda^2\frac{1-a\lambda^{\gamma-1}}{q_a(0)^2(\gamma-1)}}{\lambda^3(1-a\lambda^{\gamma-1})^2}
\end{aligned}
$$

which can be expanded

$$
\begin{aligned}
&=\frac{2\,(1-\lambda/q_a(0))}{\lambda^3}\frac{1}{a(\gamma-1)\lambda^{\gamma-2}}\frac{d}{d\lambda}\frac{1}{1-a\lambda^{\gamma-1}}-\frac{2}{q_a(0)^2(\gamma-1)\lambda(1-a\lambda^{\gamma-1})}\\
&=\frac{2\,(1-\lambda/q_a(0))}{\lambda^3 a(\gamma-1)\lambda^{\gamma-2}}\frac{d}{d\lambda}\left(\sum_{n=0}^{Q}(a\lambda^{\gamma-1})^n+\frac{(a\lambda^{\gamma-1})^{Q+1}}{1-a\lambda^{\gamma-1}}\right)\\
&\quad-\frac{2}{q_a(0)^2(\gamma-1)\lambda(1-a\lambda^{\gamma-1})}\\
&=\frac{2\,(1-\lambda/q_a(0))}{\lambda^3}\left(\sum_{n=1}^{Q}n(a\lambda^{\gamma-1})^{n-1}+\frac{(Q+1)(a\lambda^{\gamma-1})^Q}{1-a\lambda^{\gamma-1}}+\frac{(a\lambda^{\gamma-1})^{Q+1}}{(1-a\lambda^{\gamma-1})^2}\right)\\
&\quad-\frac{2}{q_a(0)^2(\gamma-1)\lambda(1-a\lambda^{\gamma-1})}\\
&=\frac{2\,(1-\lambda/q_a(0))}{\lambda^3}\left(\sum_{n=0}^{Q-1}(n+1)(a\lambda^{\gamma-1})^n+\frac{(Q+1)(a\lambda^{\gamma-1})^Q}{1-a\lambda^{\gamma-1}}+\frac{(a\lambda^{\gamma-1})^{Q+1}}{(1-a\lambda^{\gamma-1})^2}\right)\\
&\quad-\frac{2}{q_a(0)^2(\gamma-1)\lambda(1-a\lambda^{\gamma-1})}:=\hat{\mu}_2(\lambda)
\end{aligned}
$$

for any integer $Q \geq 1$. Even more, the factor involving Q has the same value for any $Q \geq 1$. Letting $Q-1$ be N or M respectively, we obtain that

$$
\begin{aligned}
\hat{\mu}_2(\lambda) = & \frac{2}{\lambda^3}\left(\sum_{n=0}^{N}(n+1)(a\lambda^{\gamma-1})^n + \frac{(N+2)(a\lambda^{\gamma-1})^{N+1}}{1-a\lambda^{\gamma-1}} + \frac{(a\lambda^{\gamma-1})^{N+2}}{(1-a\lambda^{\gamma-1})^2}\right) \\
& - \frac{2/q_a(0)}{\lambda^2}\left(\sum_{n=0}^{M}(n+1)(a\lambda^{\gamma-1})^n + \frac{(M+2)(a\lambda^{\gamma-1})^{M+1}}{1-a\lambda^{\gamma-1}} + \frac{(a\lambda^{\gamma-1})^{M+2}}{(1-a\lambda^{\gamma-1})^2}\right) \\
& - \frac{2}{\lambda q_a(0)^2(\gamma-1)}\left(\sum_{n=0}^{P}(a\lambda^{\gamma-1})^n + \frac{(a\lambda^{\gamma-1})^{P+1}}{1-a\lambda^{\gamma-1}}\right) \\
=& 2\left(\frac{1}{\lambda^3}\sum_{n=0}^{N}(n+1)(a\lambda^{\gamma-1})^n - \frac{a^{1/(\gamma-1)}}{\lambda^2}\sum_{n=0}^{M}(n+1)(a\lambda^{\gamma-1})^n\right. \\
& \left. - \frac{a^{2/(\gamma-1)}}{\lambda(\gamma-1)}\sum_{n=0}^{P}(a\lambda^{\gamma-1})^n\right) + \frac{2}{1-a\lambda^{\gamma-1}}\left(\frac{(N+2)(a\lambda^{\gamma-1})^{N+1}}{\lambda^3}\right. \\
& \left. - \frac{a^{1/(\gamma-1)}(M+2)(a\lambda^{\gamma-1})^{M+1}}{\lambda^2} - \frac{a^{2/(\gamma-1)}(a\lambda^{\gamma-1})^{P+1}}{\lambda(\gamma-1)}\right) \\
& + \frac{2}{(1-a\lambda^{\gamma-1})^2}\left(\frac{(a\lambda^{\gamma-1})^{N+2}}{\lambda^3} - \frac{a^{1/(\gamma-1)}(a\lambda^{\gamma-1})^{M+2}}{\lambda^2}\right) \\
& := r1(\lambda) + r2(\lambda) + r3(\lambda)
\end{aligned}
$$

Again, the function $\hat{\mu}_2$ is not defined at $\lambda = a^{-1/(\gamma-1)}$. Remember that $q_a(0) = a^{-1/(\gamma-1)}$. Since [3]

$$
\begin{aligned}
\lim_{\lambda\to a^{-\frac{1}{\gamma-1}}}\hat{\mu}_2(\lambda) = & \lim_{\lambda\to a^{-\frac{1}{\gamma-1}}}\frac{2\left(1-\lambda/q_a(0)-\lambda^2\frac{1-a\lambda^{\gamma-1}}{q_a(0)^2(\gamma-1)}\right)}{\lambda^3(1-a\lambda^{\gamma-1})^2} \\
= & \lim_{\lambda\to a^{-\frac{1}{\gamma-1}}}\frac{2\left(-1/q_a(0)-2\lambda\frac{1-a\lambda^{\gamma-1}}{q_a(0)^2(\gamma-1)}+\frac{a\lambda^{\gamma}}{q_a(0)^2}\right)}{3\lambda^2(1-a\lambda^{\gamma-1})^2-2(1-a\lambda^{\gamma-1})a(\gamma-1)\lambda^{\gamma+1}} \\
= & \lim_{\lambda\to a^{-\frac{1}{\gamma-1}}}\frac{2\left(2\frac{a\lambda^{\gamma-1}}{q_a(0)^2}+\frac{a\gamma\lambda^{\gamma-1}}{q_a(0)^2}\right)}{2a^2\lambda^{2\gamma-1}(\gamma-1)^2} = \frac{\frac{2+\gamma}{q_a(0)^2}}{a^2(\gamma-1)^2a^{-\frac{1}{\gamma-1}(2\gamma-1)}} \\
= & \frac{2+\gamma}{(\gamma-1)^2}a^{\frac{2}{\gamma-1}-2+\frac{2\gamma-1}{\gamma-1}} = \frac{2+\gamma}{(\gamma-1)^2}a^{\frac{2-2(\gamma-1)+2\gamma-1}{\gamma-1}} = \frac{2+\gamma}{(\gamma-1)^2}a^{\frac{3}{\gamma-1}},
\end{aligned}
$$

the function $\hat{\mu}_2$ is analytically extendable to the slit plane, $\mathbb{C}\setminus(-\infty,0]$. Since $r1$ is analytic in the slit plane, so is the extension of $r2+r3$.

Let $R \geq 0$. Pick integers M, N, P such that

$$
\begin{aligned}
(P+1)(\gamma-1) > & R \geq P(\gamma-1) \\
(M+1)(\gamma-1) > & R+1 \geq M(\gamma-1) \\
(N+1)(\gamma-1) > & R+2 \geq N(\gamma-1).
\end{aligned}
$$

[3] In the third line I omitted the terms that are going to be zero.

Then $(d/d\lambda)^R(r1(\lambda))$ inverts to

$$(-1)^R t^R \left(\sum_{n=0}^{N} \frac{2t^2(n+1)}{\Gamma(3+n(1-\gamma))}(at^{1-\gamma})^n - \sum_{n=0}^{M} \frac{2a^{1/(\gamma-1)}t(n+1)}{\Gamma(2+n(1-\gamma))}(at^{1-\gamma})^n \right. \\ \left. - \sum_{n=0}^{P} \frac{2a^{2/(\gamma-1)}}{(\gamma-1)\Gamma(1+n(1-\gamma))}(at^{1-\gamma})^n \right)$$

Next, we will show that the inverse of the remaining terms, that is, the inverse of $(\frac{d}{d\lambda})^R(r2+r3)$, converges to zero as the argument gets larger. So we have to show that the extension of $\lambda(\frac{d}{d\lambda})^R(r2+r3)$ is analytic and bounded in a sectorial region containing the right halfplane, as well as converges to zero as λ goes to zero. We already showed that the analyticity condition is satisfied. Thus, all we have to control is the behaviour at infinity and at zero. Now,

$$\begin{aligned} \left(\frac{d}{d\lambda}\right)^R r3(\lambda) &= \left(\frac{d}{d\lambda}\right)^R \frac{\left(2(a\lambda^{\gamma-1})^{N+1} - 2a^{1/(\gamma-1)}\lambda(a\lambda^{\gamma-1})^{M+1}\right)}{\lambda^2} \frac{a\lambda^{\gamma-2}}{(1-a\lambda^{\gamma-1})^2} \\ &= \sum_{i=0}^{R}(c_i\lambda^{(\gamma-1)(N+1)-2-i} - d_i\lambda^{(\gamma-1)(M+1)-1-i}) \left(\frac{d}{d\lambda}\right)^{R-i} \frac{a\lambda^{\gamma-2}}{(1-a\lambda^{\gamma-1})^2} \\ &= \sum_{i=0}^{R}(c_i\lambda^{(\gamma-1)(N+1)-2-i} - d_i\lambda^{(\gamma-1)(M+1)-1-i}) \left(\frac{d}{d\lambda}\right)^{R-i+1} \frac{1/(\gamma-1)}{1-a\lambda^{\gamma-1}} \end{aligned}$$

Using Lemma 2, we obtain for $|\lambda| \to \infty$,

$$\lambda \left(\frac{d}{d\lambda}\right)^R r3(\lambda) = O(|\lambda|^{1+(\gamma-1)(N+1)-2+1-\gamma-(R+1)}) \\ + O(|\lambda|^{1+(\gamma-1)(M+1)-1+1-\gamma-(R+1)}) = O(|\lambda|^{(\gamma-1)N-2-R}) + O(|\lambda|^{(\gamma-1)M-1-R})$$

For $\lambda \to 0$

$$\begin{aligned} \lambda \left(\frac{d}{d\lambda}\right)^R r3(\lambda) &= O(|\lambda|^{1+(\gamma-1)(N+1)-2+\gamma-1-(R+1)}) \\ &\quad + O(|\lambda|^{1+(\gamma-1)(M+1)-1+\gamma-1-(R+1)}) \\ &= O(|\lambda|^{(\gamma-1)(N+2)-(R+2)}) + O(|\lambda|^{(\gamma-1)(M+2)-(R+1)}). \end{aligned}$$

Hence, $\lambda\left(\frac{d}{d\lambda}\right)^R r3(\lambda)$ stays bounded as $|\lambda| \to \infty$ and converges to zero as $\lambda \to 0$.

The function $\lambda(d/d\lambda)^R r2(\lambda)$ has the same properties, since

$$\left(\frac{d}{d\lambda}\right)^R r2(\lambda) = \sum_{i=0}^{R} \left(c_i\lambda^{(\gamma-1)(N+1)-3-i} - d_i\lambda^{(\gamma-1)(M+1)-2-i} \right. \\ \left. -e_i\lambda^{(\gamma-1)(P+1)-1-i}\right) \left(\frac{d}{d\lambda}\right)^{R-i} \frac{1}{1-a\lambda^{\gamma-1}},$$

which implies that for $|\lambda| \to \infty$,

$$\begin{aligned}\lambda \left(\frac{d}{d\lambda}\right)^R r2(\lambda) =& O(\lambda^{1+(\gamma-1)(N+1)-3+1-\gamma-R}) + O(\lambda^{1+(\gamma-1)(M+1)-2+1-\gamma-R}) \\ &+ O(\lambda^{1+(\gamma-1)(P+1)-1+1-\gamma-R}) \\ =& O(\lambda^{N(\gamma-1)-(R+2)}) + O(\lambda^{M(\gamma-1)-(R+1)}) + O(\lambda^{P(\gamma-1)-R}).\end{aligned}$$

For $\lambda \to 0$ we write $r2(\lambda) = r20(\lambda) + r21(\lambda)$ where $r20(\lambda)$ contains the terms with $i < R$ and $r21(\lambda)$ contains the terms with $i = R$. Then we obtain that

$$\begin{aligned}&\lambda \left(\frac{d}{d\lambda}\right)^R r20(\lambda) = O(\lambda^{1+(\gamma-1)(N+1)-3-i+\gamma-1-(R-i)}) \\ &+ O(\lambda^{1+(\gamma-1)(M+1)-2-i+\gamma-1-(R-i)}) + O(\lambda^{1+(\gamma-1)(P+1)-1-i+\gamma-1-(R-i)}) \\ =& O(\lambda^{(N+1)(\gamma-1)-(R+3-\gamma)}) + O(\lambda^{(M+1)(\gamma-1)-(R+2-\gamma)}) + O(\lambda^{(P+1)(\gamma-1)-(R+1-\gamma)})\end{aligned}$$

while

$$\begin{aligned}\lambda \left(\frac{d}{d\lambda}\right)^R r21(\lambda) =& O(\lambda^{(N+1)(\gamma-1)-(R+2)}) \\ &+ O(\lambda^{(M+1)(\gamma-1)-(R+1)}) + O(\lambda^{(P+1)(\gamma-1)-R})\end{aligned}$$

so that $r21(\lambda)$ dominates. Hence $\lambda \left(\frac{d}{d\lambda}\right)^R (r2+r3)(\lambda)$ is bounded in a sectorial region containing the right halfplane and converges to zero as λ goes to zero. Thus the Laplace inverse of $\left(\frac{d}{d\lambda}\right)^R (r2+r3)(\lambda)$, say $g(t)$ converges to zero as t goes to infinity. Hence

$$\begin{aligned}&(-t)^R \mu_2(t) = (-t)^R \left(\sum_{n=0}^{N} \frac{2t^2(n+1)}{\Gamma(3+n(1-\gamma))} (at^{1-\gamma})^n \right. \\ &\left. - \sum_{n=0}^{M} \frac{2a^{1/(\gamma-1)} t(n+1)}{\Gamma(2+n(1-\gamma))} (at^{1-\gamma})^n \sum_{n=0}^{P} \frac{2a^{2/(\gamma-1)}}{(\gamma-1)\Gamma(1+n(1-\gamma))} (at^{1-\gamma})^n \right) + g(t)\end{aligned}$$

or

$$\begin{aligned}\mu_2(t) =& \sum_{n=0}^{N} \frac{2t^2(n+1)}{\Gamma(3+n(1-\gamma))} (at^{1-\gamma})^n - \frac{2a^{1/(\gamma-1)} t(n+1)}{\Gamma(2+n(1-\gamma))} (at^{1-\gamma})^n \\ &- \frac{2a^{2/(\gamma-1)}}{(\gamma-1)\Gamma(1+n(1-\gamma))} (at^{1-\gamma})^n + o(t^{2-N(\gamma-1)})\end{aligned}$$

Hence we showed (12) for all $N \le (R+2)/(\gamma-1)$. Since R was chosen arbitrarily, the theorem is proven. □

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