From Fractals to Fractional Vector Calculus: Measurement in the Correct Metric

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Traditional (stationary) stochastic theories have been fairly successful in reproducing transport behavior at relatively homogeneous field sites such as the Borden and Cape Code sites. However, the highly heterogeneous MADE site has produced tracer data that can not be adequately explained with traditional stochastic theories. In recent years, considerable attention has been focused on developing more sophisticated theories that can predict or reproduce the behavior of complex sites such as the MADE site. People began to realize that the model for geologic complexity may in many cases be very different than the model required for stochastic theory. Fractal approaches were useful in conceptualizing scale-invariant heterogeneity by demonstrating that scale dependant transport was just an artifact of our measurement system. Fractal media have dimensions larger than the dimension that measurement is taking place in, thus assuring the scale-dependence of parameters such as dispersivity. What was needed was a rigorous way to develop a theory that was consistent with the fractal dimension of the heterogeneity. The fractional advection-dispersion equation (FADE) was developed with this idea in mind. The second derivative in the dispersion term of the advection-dispersion equation is replaced with a fractional derivative. The order of differentiation, α , is fractional. Values of α in the range: $1 < \alpha < 2$ produce super-Fickian dispersion; in essence, the dispersion scaling is controlled by the value of α. When $\alpha = 2$, the traditional advection-dispersion equation is recovered. The 1-D version of the FADE has been used successfully to back-predict tracer test behavior at several heterogeneous field sites, including the

MADE site. It has been hypothesized that the order of differentiation in the FADE is equivalent to (or at least related to) the fractal dimension of the geologic heterogeneity. With this way of thinking, one can think of the FADE as a governing equation written for the correct dimension, thus eliminating scale-dependent behavior. Before a generalized multi-dimensional form of the FADE can be developed, it has been necessary to develop a generalized fractional vector calculus. The authors have recently developed generalized canonical fractional forms of the gradient, divergence and curl. The physical meaning and consequences of the fractional gradient and divergence is explored. In a highly heterogeneous aquifer, it is shown that the fractional gradient of the head can cause flow due to a perturbation of the head at a distance. The fractional gradient can also cause flow in a direction that is different from what would be expected with the usual integer order gradient. The fractional divergence operator represents conservation of mass for a highly heterogeneous aquifer. Such aquifers will have power law velocity distributions with velocities ranging over many orders of magnitude. The fractional divergence is able to conserve mass by allowing fluids to move into (or out of) a control volume from non-adjacent volumes. Traditional divergence only allows mass to move to/from adjacent faces of a control volume, which is inadequate for highly heterogeneous aquifers.