Stochastic Processes of Variable Fractional Order: Dirichlet Forms and Feller Processes

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We are interested in various forms of variable-order non-local operators which give rise to stochastic processes. There are several possibilities to introduce such operators; the most popular is, probably, the *stable-like* generator $-(-\Delta)^{\alpha(x)}$ considered by R. Bass et al. Our approach is to consider (generalized) stable-like Dirichlet forms of the following type

$$\mathcal{E}(u,v) = \iint (u(y)-u(x))(v(y)-v(x))\,\frac{dx\,dy}{|x-y|^{n+\alpha(x,y)}}$$

where $\alpha(x,y)$ is a function on $\mathbb{R}^n \times \mathbb{R}^n$ and values in $\mathbb{R}$. If $\alpha(x,y) = \alpha(x) \in (0,2]$ we are, essentially, in the stable-like setting. In this talk we address the following points

- *Existence of an associated stochastic process.* For this we use the theory of Dirichlet forms and give necessary and sufficient criteria in terms of $\alpha(x,y)$.

- *Structure of the infinitesimal generator and its symbol.* In general, it is very hard to calculate the explicit form of the generator of a Dirichlet form. In our case it turns out that under certain integrability assumptions on the kernel $|x-y|^{-\alpha(x,y)-n}$ the generator is a pseudo-differential operator with negative definite symbol $p(x,\xi)$. From the symbol it is easily possible to read off the complete semimartingale characteristics of the associated jump-diffusion (in the sense of Jacod-Shiraev).

- *Probability estimates.* Using the symbol we obtain estimates for the first passage times σ_r^x of the associated stochastic process from a ball $B_r(x)$. Typically, such estimates look like

$$c \Big/ \sup_{|x-y|\le r} \sup_{|z|=1} \operatorname{Re} p(y,z/r) \le \mathbb{E}^x \sigma_r^x \le C \Big/ \inf_{|x-y|\le r} \sup_{|z|=1} \operatorname{Re} p(y,z/r).$$

- *Existence of a good version of the process.* A major problem in the theory of Dirichlet forms is the existence of capacity-zero exceptional sets. Combining the above probability estimates with techniques due to Bass and Levin, Song and Vondracek, and Bass and Kaßmann we are able to show that certain $\mathcal{E}$-harmonic functions are (Hölder-)continuous. This can then be used to construct a Feller version of the Resolvent and the associated semigroup. This allows us to pick a 'good', i.e. Feller-version of the process without any exceptional sets. Using perturbation techniques one sees that our arguments are still valid if we change the kernel $|x-y|^{-n-\alpha(x,y)}$ whenever $|x-y| > \epsilon$, i.e. off the diagonal.

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