The Mathematical Statistics of a Second-Order Stationary Process, with Potential Applications to Permeability (k) of Heterogeneous Sediments and Velocity (v) in Turbulent Flows

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The theory of non-stationary stochastic processes with stationary increments, taken over the lag "h", gives rise to stochastic fractals. When such fractals are used to represent measurements of (assumed stationary) physical properties, such as $ln(k)$ increments "$\Delta_h ln(k)$" in sediments or velocity increments "$\Delta_h v$" in turbulent flows, the resulting measurements exhibit scaling, either spatial, temporal or both. (In the present context, such scaling refers to systematic changes in the statistical properties of the increment distributions, such as variance, with the lag size over which the increments are determined.) Depending on the class of probability density functions (PDFs) that describe the property increment distributions, the resulting stochastic fractals will display different mathematical behavior. Until recently, the stationary increment process was represented using mainly Gaussian, Gamma or Levy PDFs. However, measurements in both sediments and fluid turbulence indicate that these PDFs are not commonly observed. Based on recent data and previous studies referenced and discussed in Meerschaert et al. (2004) and Molz et al. (2005), the measured increment PDFs display an approximate double exponential (Laplace) shape at smaller lags, and this shape evolves towards Gaussian at larger lags (Frisch, 1995; van de Water, 1998). A model for this behavior developed from the uncorrelated Generalized Laplace PDF family (Kotz et al., 2001) called fractional Laplace motion, in analogy with its Gaussian counterpart-fractional Brownian motion, has been suggested (Meerschaert et al., 2004) and the necessary mathematics elaborated (Kozubowski et al., 2005). The resulting stochastic fractal is not a typical self-affine monofractal, but it does exhibit monofractal-like scaling in certain lag size ranges. To date, it has been shown that the autocorrelated Generalized Laplace family fits $ln(k)$ increment distributions and reproduces the Gaussian limit of Kolmolgorov's original 1941 theory (Frisch, 1995) when applied to Eulerian turbulent velocity increments. However, to make a mathematically and physically self-consistent application to turbulence, one must adopt a Lagrangian viewpoint, which requires conceptualizing velocity statistics along particular particle trajectories. An interesting development resulting from this viewpoint is that a second differencing is required to get from particle position (non-stationary process) to velocity (first differencing–stationary process) to velocity increments (second differencing–another stationary process). As stated previously, the non-stationary process is called fractional Laplace motion (fLam), the first-order stationary process is called fractional Laplace noise (fLan), and the second-order stationary process is the noise of fractional Laplace noise (fLann), which is represented by another new PDF family. To date, closed-form expressions for the even order moments of the underlying fLann PDFs have been developed, and other properties are currently being studied. A potential analogy between turbulent $\Delta_h v$ and sediment $\Delta_h ln(k)$ is intriguing, and perhaps offers insight into the underlying chaotic processes that constitute turbulence and may result also in the pervasive heterogeneity observed in most natural sediments (Frisch, 1995; Faybishenko, 2004). Properties of fLann are presented, and potential applications to both sediments and fluid turbulence are illustrated and discussed.

References

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