Numerical Methods for Fractional-Order PDEs

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I will describe two approaches to the numerical solution of fractional-order initial-value problems of the form

$$\frac{\partial u}{\partial t} + \left(\frac{\partial}{\partial t}\right)^{-\alpha} Au = f(t) \quad \text{for } t > 0, \text{ with } u(0) = u_0,$$

for $-1 < \alpha < 1$. In practice, the linear operator A is a strongly elliptic, partial differential operator in some spatial variables $x = (x_1, \ldots, x_d)$; for instance, in the simplest case, $d = 1$ and $Au = -\partial^2 u/\partial x^2$.

The first approach is a two-level time-stepping scheme that generalises the classical Crank–Nicolson method for the heat equation. The scheme is formally second-order accurate, but due to the singular behaviour of the derivatives of u as t tends to 0 it is necessary to use a non-uniform time step k_n to achieve $O(k^2)$ accuracy, where $k = \max_n k_n$. This error bound was proved in joint work with Kassem Mustapha.

The second approach is to apply a quadrature rule to approximate the contour integral in the Laplace inversion formula,

$$u(t) = \frac{1}{2\pi i} \int_\Gamma e^{zt} \hat{u}(z)\, dz,$$

where $\hat{u}(z)$ is (an analytic continuation of) the Laplace transform of $u(t)$. By deforming the contour of integration Γ so that $\mathrm{Re}(z) \to -\infty$ as $|\,\mathrm{Im}(z)| \to \infty$ for z on Γ, and by taking an appropriate parametric representation $z(\xi)$ the integrand exhibits a double exponential decay like $e^{-ct\cosh(\xi)}$, for a constant $c > 0$, as $\xi \to \pm\infty$. If t is bounded away from zero, then a very simple choice of an N-point quadrature rule leads to convergence of order $O(e^{-c\log N/N})$, and in some cases $O(e^{-cN})$. For a difference choice of quadrature one can achieve $O(e^{-c\sqrt{N}})$ convergence uniformly for $0 \le t \le T$. Methods of this type, mainly for classical parabolic problems ($\alpha = 0$), have received much attention in recent years, not only because of their high accuracy but also because of their naturally parallel structure. I will describe some joint work with Ian Sloan and Vidar Thomée.