Power Laws in Continuous Time Random Walks and the Distinguished Role of the Mittag-Leffler Waiting Time Density

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The entire transcendental function $E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1-k\beta)}$, introduced in the year 1903 by G. Mittag-Leffler and named after him (see [3] for a good survey of its properties) and some of its generalizations in recent two decades gradually have come to be recognized as useful in diverse applications. Some such applications have been discovered at first only in the Laplace transform domain, see [1] and [4], the authors being unaware of the fact that in the time domain they reduce to the closely related functions that we take into view in this lecture, namely, for $0 < \beta \leq 1, t > 0$,the variants $\Psi_\beta(t) = E_\beta(-t^\beta)$ and $\Psi_\beta(t) = -\frac{d}{dt}\Psi_\beta(t)$.

Both functions are completely monotone and appear in the standard process of fractional relaxation (see e.g. [5]), the modelling being done by the Caputo or by the Riemann-Liouville fractional derivative. They also play decisive roles as waiting time probability distribution or density in the theory of continuous time random walks (see e.g. [7], [8], [10]) and appear as the limiting probability law in infinite thinning of a renewal process for an initial density with power law asymptotics (see [4]). We outline these models and show furthermore that the Mittag-Leffler waiting time law is the precise description of the properly rescaled long time behaviour of a renewal process whose waiting time density has a long tail due to a power law decay at infinity. In this sense the Mittag-Leffler process, generalizing the classical Poisson process and described in [9] and [10], is an important limiting process. In the analysis of such asymptotic behaviour the Mittag-Leffler density $\Psi_\beta(t)$ exhibits stability against a rescaling procedure combined with a deceleration and a characteristic kind of self-similarity. Remarkably there arises the same transformation formula in the analysis of infinite thinning and of long time behaviour of a power law renewal process. It is instructive to consider some variants of passing to the diffusion limit (limit in space only or in time only or in both simultaneously, the time-fractional drift process) and to pay attention to distributed fractional order processes [2] and and their approximation by multiply scaled continuous time random walks [6].

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