

# On the number of cycles in 3-connected cubic graphs

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## Abstract

*Let  $f(n)$  be the minimum number of cycles present in a 3-connected cubic graph on  $n$  vertices. In 1986, Barefoot, Clark and Entringer showed that  $f(n)$  is sub-exponential and conjectured that  $f(n)$  is super-polynomial. We verify this by showing that, for  $n$  sufficiently large,*

$$2^{n^{0.17}} < f(n) < 2^{n^{0.95}}.$$

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## 1. Introduction.

In [1], Barefoot, Clark and Entringer showed that a 2-connected cubic graph on  $n$  vertices contains at least  $(n^2+14n)/8$  cycles and that this bound is realized by a class of cubic graphs of connectivity 2. In the same paper, it was conjectured that,  $f(n)$  is super-polynomial (see also [3]). This would indicate a striking difference between the 2-connected and 3-connected cubic graphs. While this increase is dramatic, it is not unexpected when one considers the ladders (see Figure 1). The lower bound on the number of cycles in 2-connected cubic graphs is realized by cubic versions of the ladder, while the obvious 3-connected variations of this structure, the extended ladder and the Möbius ladder, have exponentially many cycles. It is possible that by requiring that our graphs be cyclically 4-edge-connected, the number of cycles may grow exponentially with  $n$ .

## 2. Preliminary results.

We begin our treatment by establishing the upper bound for  $f(n)$ . This may be done immediately from the following results.

**Theorem 2.1.** [2] *For each even natural number  $n$ , there exists a 3-connected cubic graph,  $G$  on  $n$  vertices such that the longest cycle in  $G$  has length at most  $cn^{\log 8/\log 9}$  for some constant  $c$ .* ■

**Corollary 2.2.** [1] *For each even natural number  $n$ , there exists a 3-connected cubic graph on  $n$  vertices with fewer than  $n2^{cn^{\log 8/\log 9}}$  cycles for some constant  $c$ .* ■

Since  $\log 8/\log 9 \simeq 0.946395$ , we deduce that for  $n$  sufficiently large,  $f(n) < 2^{n^{0.95}}$ , establishing our upper bound.

To determine the lower bound for  $f(n)$ , we again turn our attention to a longest cycle in a 3-connected cubic graph; this time from the aspect of the minimum length such a cycle must have. We shall make use of the following result of Jackson's.

**Theorem 2.3.** [4] *Let  $G$  be a 3-connected cubic graph on  $n$  vertices, and let  $e_1, e_2 \in E(G)$ . Then  $e_1$  and  $e_2$  are contained in a cycle of length at least  $n^t + 1$ , where  $t = \log_2(1 + \sqrt{5}) - 1$  ( $\simeq 0.69$ ).* ■

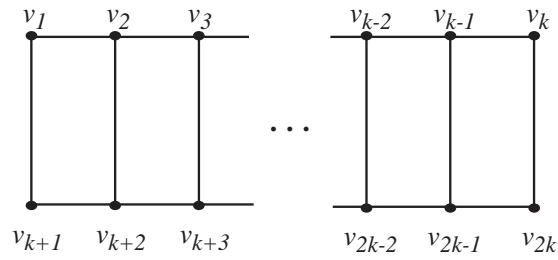
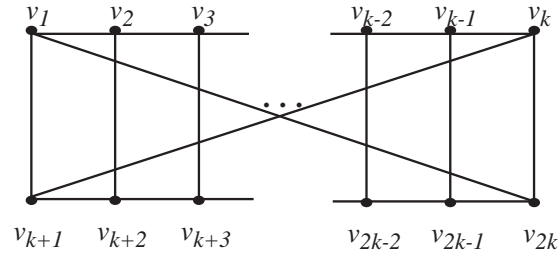
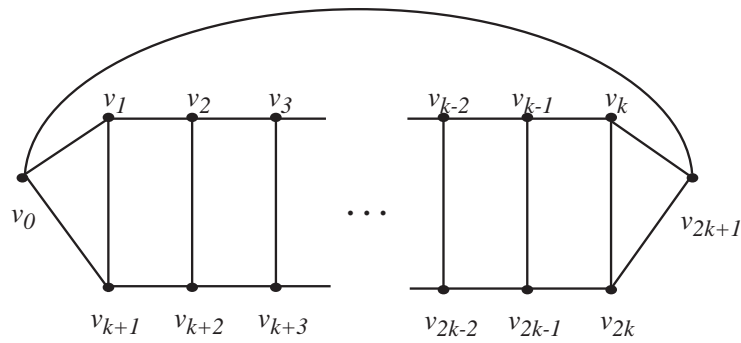
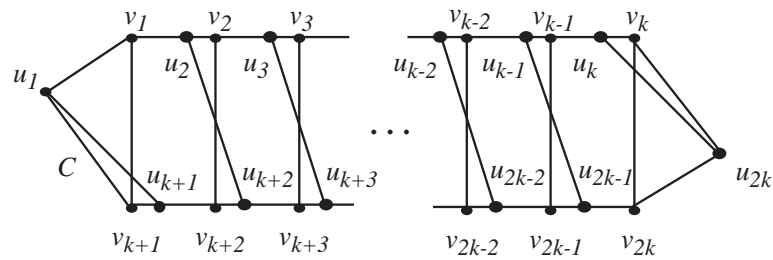
The ladder on  $2k$  verticesThe Möbius ladder  $ML_{2k}$ The extended ladder  $L_{2k}$ A cross-ladder (on a cycle  $C$ ) with  $k$  crosses

Figure 1. Ladder based graphs.

Working from a longest cycle, we will produce substructures in which the number of cycles is bounded below. Among these substructures are the Möbius ladders, the extended ladders and the cross-ladders (defined pictorially in Figure 1.). In a

subdivision  $H$  of the cross-ladder based on a cycle  $C$ , the segments of  $C$  between consecutive pairs of crossing chords are called *inter-cross-segments*. A *bypass* to  $H$  is a path with both end vertices in the same inter-cross-segment and no other vertices in common with  $H$ . A set,  $B$ , of pairwise disjoint bypasses to  $H$  is said to be *clear* if for each bypass,  $b \in B$ , that part of the inter-cross-segment between the end vertices of  $b$  contains no end vertex of a bypass in  $B \setminus \{b\}$ . Some bounds on the numbers of cycles in these various ladders are included in the following lemmas.

**Lemma 2.4.** *The Möbius ladder,  $ML_{2k}$  contains at least  $2^k$  cycles.* ■

**Lemma 2.5.** *The extended ladder,  $L_{2k}$  contains at least  $2^k$  cycles.* ■

**Lemma 2.6.** *The cross-ladder with  $k$  crosses and a clear set of  $j$  bypasses contains at least  $2^{j+k}$  cycles. (Note that  $j = 0$  and/or  $k = 0$  are possible.)* ■

Finally in this section, we include two results which will prove very useful in determining the existence of some desired substructures. The first is a classical theorem due to Erdős and Szekeres and the other a technical observation.

**Theorem 2.7.** [5] *Let  $S$  be a sequence of  $jk$  distinct positive integers, where  $j$  and  $k$  are both positive integers. Then  $S$  contains either a subsequence of length  $j$  which is monotonically increasing or a subsequence of length  $k$  which is monotonically decreasing.* ■

**Observation 2.8.** *Let  $G$  be a connected graph with maximum degree at most 3 and let  $S$  be the set of vertices of degree 2 in  $G$ . Then  $G$  contains pairwise disjoint paths  $p_1, p_2, \dots, p_{\lfloor \frac{|S|}{2} \rfloor}$  whose endvertices form a subset of  $S$ .*

**Proof.** Suppose, by way of a contradiction, that  $G$  is a connected graph of maximum degree at most 3 for which no such set of paths exists and assume that  $P = \{p_1, p_2, \dots, p_k\}$  is a largest possible set of disjoint paths each joining a pair of vertices in  $S$ . Let  $S'$  denote the set of end vertices of paths in  $P$ . By our choice of  $G$ , there are vertices  $u, v \in S \setminus S'$  and since  $G$  is connected, there is a path  $p'$  in  $G$  joining  $u$  to  $v$ . The subgraph of  $G$  induced by  $(\cup_{i=1}^k E(p_i)) \Delta E(p')$  (where  $\Delta$  denotes the symmetric difference) consists of paths and cycles. The set of paths  $P'$  in this subgraph has as its set of end vertices,  $S' \cup \{u, v\}$ , contradicting our choice of  $P$ . This contradiction completes the proof of the observation. ■

### 3. The main result.

We are now ready to present our main theorem.

**Theorem 3.1.** *Let  $f(n)$  denote the minimum number of cycles in a 3-connected cubic graph on  $n$  vertices. Then, for  $n$  sufficiently large,  $2^{n^{0.17}} < f(n) < 2^{n^{0.95}}$ .*

**Proof.** In the previous section we established the upper bound for  $f(n)$  stated in the theorem so we concern ourselves here only with the lower bound. Let  $G$  be a 3-connected cubic graph on  $n$  vertices with as few cycles as possible. By Theorem 2.3,  $G$  contains a longest cycle  $C$  of length at least  $n^t + 1$ , where  $t = \log_2(1 + \sqrt{5}) - 1$ . We construct from  $G$  a Hamiltonian cubic graph,  $G'$ , with a spanning cycle,  $C'$ , derived from  $C$ . For  $u, v \in V(C)$  with  $uv \in E(G) \setminus E(C)$ , include  $u, v \in V(G')$  and  $uv \in E(G')$  as a chord to  $C'$ . For vertices  $w, x, y \in V(C)$  such that  $\{w, x, y\} = N(z)$ ,  $z \in V(G) \setminus V(C)$ , delete  $z$ , “suppress” the vertex  $w$ , include  $x, y \in V(G')$  and  $xy \in E(G')$  as a chord to  $C'$ . By Observation 2.8, for each non-trivial component of  $G - V(C)$ ,  $K$ , say, there are pairwise disjoint paths through  $K$  joining, in pairs, all but at most one of the vertices of attachment for  $K$  in  $C$ . In the event that one vertex of attachment is missed, suppress it and replace the paths through  $K$  by chords between the same pairs of end vertices on the cycle  $C'$  obtained from  $C$  after all such suppressions of vertices have been carried out.

In the process above, at most one third of the vertices in  $C$  are suppressed and thus the resulting Hamiltonian cycle,  $C'$ , contains at least  $\frac{2}{3}(n^t + 1)$  vertices. Let  $|V(C')| = 2m^4$ . We shall prove that  $G$  contains at least  $2^m$  cycles.

First, label the vertices in  $C'$ ,  $x_0, x_1, \dots, x_{(2m^4-1)}$  in order, travelling clockwise around  $C'$ . With this labelling imposed, we consider  $C'$  to be divided into  $m$  equal segments,  $[x_0 \dots x_{(2m^3-1)}]$  and so on. If each of the segments contains both end vertices of a chord to  $C'$ , then we may travel from one end of the segment to the other in either of two distinct ways without using vertices from any other segments. In this way, we can construct at least  $2^m$  cycles in  $G$ .

So let us assume that the segment  $[x_0 \dots x_{(2m^3-1)}] = S_1$ , say, does not contain both end vertices of a chord to  $C'$ . Now each vertex  $x_i \in S_1$  has a different neighbour,  $x_{\alpha(i)} \in V(C') \setminus S_1$ . By Theorem 2.7, the sequence  $\alpha(0), \alpha(1), \dots, \alpha(2m^3 - 1)$  contains either a monotonically increasing subsequence of length at least  $m$  or a monotonically decreasing subsequence of length at least  $2m^2$ .

In the former case,  $G$  contains a subgraph which is isomorphic to a subdivision of the Möbius ladder,  $ML_{2m}$  and hence  $G$  contains at least  $2^m$  cycles.

In the latter case, let the chords of  $C'$  defined by the monotonically decreasing sequence be labelled  $c_1, c_2, \dots, c_{2m^2}$  from left to right. To facilitate the discussion which follows, we pull back the structure we have isolated to the corresponding subgraph of  $G$ . This subgraph,  $H$  say, consists of the cycle  $C$  together with pairwise

disjoint paths  $P_1, P_2, \dots, P_{2m^2}$  corresponding to the chords  $c_1, c_2, \dots, c_{2m^2}$ . The vertices in  $C$  joined by the path  $P_i$  are labelled  $a_i$  and  $b_i$  with the understanding that  $a_i$  corresponds to a vertex in  $S_1$ . We shall assume that  $H$  is drawn so that the segment of  $C$  corresponding to  $S_1$  is at the top while the vertices  $b_1, b_2, \dots, b_{2m^2}$  are on the bottom. Segments of  $C$  indicated by interval notation refer to the segment of  $C$  between the listed end vertices, travelling in a clockwise direction. The usual conventions of open and closed interval notation are observed.

Our graph  $G$  is 3-connected, so  $\{a_i, b_i\}$  cannot form a cutset. Thus, for each  $i = 1, 2, \dots, 2m^2$ , there is a path in  $G$  from the left hand component of  $H - \{a_i, b_i\}$  to the right hand component (note that such a path may intersect the path  $P_i$ ). If any such path “jumps” at least  $m$  of the paths  $P_1, \dots, P_{2m^2}$ , then  $G$  contains a subgraph isomorphic to a subdivision of the extended ladder  $L_{2m}$  and hence it contains at least  $2^m$  cycles. Consequently, we may assume that each such path in  $G$  “jumps” at most  $m - 1$  of the paths  $P_1, \dots, P_{2m^2}$  in  $H$ . We consider the paths  $P_1, \dots, P_{2m^2}$  to be partitioned into  $m$  parcels of consecutive paths, the first  $m - 1$  parcels each containing  $2m + 1$  paths and the last parcel containing the remaining  $m + 1$  paths. In what follows we shall construct a subdivision of a cross-ladder together with a clear set of bypasses.

As before, in no parcel can the end vertices of the middle path,  $P_i$ , form a cutset. Thus there is a path in  $G$  joining the left hand component of  $H - \{a_i, b_i\}$  to the right hand component. Furthermore, if this path has only its end vertices in  $H$ , then both of these end vertices lie between  $P_{i-m}$  and  $P_{i+m}$  (including all paths in the parcel and those vertices of  $C$  contained in segments between the paths in the parcel other than  $a_i$  and  $b_i$ ). Now, if such a path in  $G$  has one end vertex,  $x_i$  in a segment  $(a_j, a_{j+1})$  of  $C$ , and the other,  $y_i$  in  $(b_{j'+1}, b_{j'})$ , then  $P_i$  and the path in  $G$  from  $x_i$  to  $y_i$  form a subdivision of a cross to  $C$ . However else the end vertices of these paths are arranged (either both in upper segments of  $C$ , both in lower segments of  $C$ , one in an upper segment of  $C$  and the other on some path  $P_j$ , etc.) we form a bypass to the subdivision of a cross-ladder. By the construction employed, we obtain either a cross or a bypass for each of the  $m$  parcels and the set of bypasses so formed must be clear. Thus, by Lemma 2.6,  $G$  contains at least  $2^m$  cycles in this final case also.

So we see that  $G$  must contain at least  $2^m$  cycles, where  $2m^4 \geq \frac{2}{3}(n^t + 1)$  (recall that  $t \simeq 0.69$ ). Thus for sufficiently large  $n$ , we have  $m > \sqrt[4]{\frac{1}{3}n^{t/4}} > n^{0.17}$ .  $\blacksquare$

**References .**

- [1] Barefoot, C. A., Clark, L. and Entringer, R., Cubic graphs with the minimum number of cycles, *Congressus Numerantium* **53** (1986) 49-62.
- [2] Bondy, J. A. and Simonovits, M., Longest cycles in 3-connected cubic graphs, *Canad. J. Math.* **32**, No. 4 (1980) 987-992.
- [3] Bondy, J.A., Basic graph theory: paths and circuits, in : *Handbook of Combinatorics* (eds. Grötschel, M., Lovász, L. and Graham, R.), North-Holland (1995), 3-110.
- [4] Jackson, W., Longest cycles in 3-connected cubic graphs, *J. Combin. Theory, Ser. B* **41** (1986) 17-26.
- [5] Erdős, P. and Szekeres, G., A combinatorial problem in geometry , *Compositio. Math.* **2** (1935) 463-470 .