Cycles through 23 vertices in 3-connected cubic planar graphs

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Abstract

We establish that if A is a set of at most 23 vertices in a 3-connected cubic planar graph G, then there is a cycle in G containing A. This result is sharp.

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Let G be a 3-connected cubic planar graph and let $A \subseteq V(G)$. It was shown in [4] that if $|A| \leq 19$ there is a cycle C in G such that $A \subseteq V(C)$. In this paper we show that if $|A| \leq 23$, then G contains a cycle through A. The sharpness of this result is apparent, as it was demonstrated in [4] that there are 3-connected cubic planar graphs in which there is a set of 24 vertices that do not lie on a common cycle.

Before proceeding we include some definitions and terminology from the existing literature, as we shall make use of them in the rest of the paper. By a *k-gon* we mean a face of a plane graph bounded by k edges. Note that a k-cycle is not necessarily a k-gon. By a *k-cut* we mean a set of k edges whose removal leaves the graph disconnected and of which no proper subset has that property. The two components (and clearly there are only two) formed by the removal of a k-cut are called *k-pieces*. A k-cut is *non-trivial* if each of its k-pieces contains a cycle and *essential* if it is non-trivial and each of its k-pieces contains more than k vertices. A cubic graph is *cyclically-k-connected* if it has no non-trivial t-cuts for $0 \leq t \leq k-1$, and has *cyclic connectivity k* if in addition it has at least one non-trivial k-cut. We denote by $\lambda'(G)$ the value of k such that the C3CP G has cyclic connectivity k.

Let G be a cubic graph and let $S = \{u_i v_i : 1 \leq i \leq 3\}$ be a non-trivial 3-cut of G. Suppose that $u, v \notin V(G)$ and that L and R are the two 3-piecess of $G - S$ with $u_i \in V(L), i = 1, 2, 3$. Then the graphs

$$H = L \cup \{u, uu_1, uu_2, uu_3\} \text{ and } J = R \cup \{v, vv_1, vv_2, vv_3\}$$

are called the *3-cut reductions* of G with respect to the 3-cut S. Note that if G is 3-connected, then so are its 3-cut reductions. If $e = xy \in E(G)$, x has neighbours $\{x_1, x_2, y\}$ and y has neighbours $\{y_1, y_2, x\}$, then the graph

$$G_e = (G - \{x, y\}) \cup \{x_1 x_2, y_1 y_2\}$$

is called the *edge reduction* of G using the edge e.

A set of vertices $A \subseteq V(G)$ is called *cyclable* if there is a cycle in G containing A. If, for each $A \subseteq V(G)$, $|A| \leq m$, there is a cycle in G containing A, then G is said to be *m-cyclable*. If A is a cyclable subset of $V(G)$ and $e \in E(G)$ is contained in every cycle in G through A, then e is said to be an *unavoidable edge given A*.

Cycles in 3-connected cubic planar graphs have been extensively studied since an early hope that all such graphs would turn out to be hamiltonian and thus provide a proof of the Four Colour Conjecture. Of course, it is now known that there are non-hamiltonian 3-connected cubic planar graphs. The smallest such graphs were determined by Holton and McKay in [7] where they proved the following result.

Theorem 1. *Every 3-connected cubic planar graph of order at most 36 is hamiltonian. Moreover, there are precisely six non-hamiltonian 3-connected cubic planar graphs of order 38.* ∎

Insisting that graphs also be cyclically 4-connected increases the likelihood of hamiltonicity but cannot guarantee a hamiltonian cycle, as is indicated by the following result from [1].

Theorem 2. *There are precisely three non-hamiltonian cyclically 4-connected cubic planar graphs of order 42 and none smaller. Furthermore, there is precisely one non-hamiltonian cyclically 5-connected cubic planar graph of order 44 and none smaller.* ∎

It has been conjectured by Barnette that all bipartite 3-connected cubic planar graphs are hamiltonian. In support of that conjecture is the following result obtained in [5].

Theorem 3. *Every 3-connected cubic planar bipartite graph of order at most 64 is hamiltonian.* ∎

In establishing the above results, it was noted that when a hamiltonian cycle exists in a 3-connected cubic planar graph, then it is frequently possible to find hamiltonian cycles which either avoid or include specified edges. Dropping the requirement of hamiltonicity and concentrating on smaller cyclable sets of vertices, the following results are proved in [4] and [2] respectively.

Theorem 4. *Let G be a 3-connected cubic planar graph and let $A \subseteq V(G)$ with $|A| \leq 9$. If $e \in E(G)$, then there is a cycle in $G - e$ containing A.* ∎

Theorem 5. *Let G is a 3-connected cubic planar graph and let $A \subseteq V(G)$ with $|A| \leq 14$. If $e \in E(G)$, then there is a cycle in G containing A and e.* ∎

We now present our main result.

Theorem 6. *Every 3-connected cubic planar graph is 23-cyclable.*

Proof. Suppose, by way of a contradiction, that the statement is false and that G is a 3-connected cubic planar graph which is not 23-cyclable. Suppose further that G is such a graph of minimum order and that A is a set of 23 vertices in G which do not lie on a common cycle. By Theorem 1 we know that G has order at least 38. We distinguish two cases.

Case 1. Assume that G contains a non-trivial 3-cut, $S = \{u_1v_1, u_2v_2, u_3v_3\}$ and let H and J be the corresponding 3-cut reductions of G. Let $A_H = A \cap V(H)$ and $A_J = A \cap V(J)$. We may assume that $|A_H| \leq |A_J|$ and thus $|A_H| \leq 11$.

(1.1) Suppose $A \subseteq V(J)$. Then, by the minimality of G, there is a cycle C_J in J containing A. If $v \notin V(C_J)$, then C_J lifts to a cycle in G through A, so we may assume that $v \in V(C_J)$. Without loss of generality, we may assume that the edge vv_1 is not in $E(C_J)$. By Theorem 4, there is a cycle C_H in H containing u and avoiding the edge uu_1. Then $(C_H - u) \cup (C_J - v)$ is a cycle in G through A.

(1.2) If $1 \le |A_H| \le 8$, then, by the minimality of G, there is a cycle C_J in J through $A_J \cup \{v\}$. Without loss of generality, C_J does not contain the edge vv_1. By Theorem 4, there is a cycle C_H in H through $A_H \cup \{u\}$ avoiding the edge uu_1. Again, $(C_H - u) \cup (C_J - v)$ is a cycle in G through A.

(1.3) Hence we may assume that $9 \le |A_H| \le 11$ and thus $|A_J| \le 14$. By Theorem 5, for each $i = 1, 2, 3$ there is a cycle in J through $A_J \cup \{vv_i\}$, and so at most one of vv_1, vv_2, vv_3 is unavoidable in J given $A_J \cup \{v\}$. Similarly, at most one of uu_1, uu_2, uu_3 is unavoidable in H given $A_H \cup \{u\}$. From this we conclude that there is a cycle in G through A.

Case 2. From the above, G must be cyclically 4-connected. If there is an edge $e = xy$ with $x, y \notin A$, then let G_e be the edge reduction of G using e (note that G_e must be 3-connected). Since $x, y \notin A$, $A \subseteq V(G_e)$. By the minimality of G, there is a cycle in G_e through A which lifts to a cycle in G through A.

Hence every edge of G is incident with a vertex of A. This means in particular that $|V(G)| \le 46$. If $|V(G)| = 46$, then G is bipartite and, by Theorem 3, hamiltonian. If $|V(G)| \le 40$, then G is hamiltonian by Theorem 2. In the case of order 42, we need only consider the three non-hamiltonian graphs indicated in Theorem 2. Each of these contains at least 24 hamiltonian vertex-deleted subgraphs and hence is 23-cyclable. Thus it remains only to consider the 44-vertex case. Here we have a near-bipartite graph in which the partition of $V(G)$ into two sets, A (as used throughout the proof to date) and $B = V(G) - A$ is such that B is an independent set and there are precisely three edges with both endvertices in A. Consequently, G has at most six odd faces. Furthermore, the minimality of G implies that G contains no pair of 4-gons with an edge in common. To see this, suppose that there is such a pair of 4-gons in G sharing the edge $e = xy$, say. If f and g are the edges opposite e in each of the 4-gons, then $G_{f,g}$, the graph obtained from G after edge reductions by f and g successively, is 3-connected cubic and planar on 40 vertices and is thus 23-cyclable. Without loss of generality, we may assume that $x \in A$ and at least two other vertices in the adjacent 4-gons must be in A. Thus $A_{f,g} = (A \cap V(G_{f,g}) \cup \{y\})$ contains at most 22 vertices and $G_{f,g}$ contains a cycle through $A_{f,g}$. It is easy to see that such a cycle lifts to the desired cycle in G through A. Using the method of Brinkmann and McKay [3], we generated all cyclically 4-connected cubic planar graphs with at most six odd faces and no pair of adjacent 4-gons. There are 8568483 such graphs. These were then checked for hamiltonicity using the method in [9]. All are hamiltonian and hence there is a cycle in G through A. This completes the proof. ∎

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