Transmission of ocean waves through a row of randomly perturbed circular ice floes

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Abstract

A method is described to compute the transmission matrix of a finite array of ice floes under ocean wave forcing. We discuss the behaviour of the transmission matrix when the floes are regularly spaced and randomly positioned. The scattering angles of the regular array are gradually filtered out under increasing disorder.

Keywords: ocean waves, directional spectra, scattering angles, random arrays

1 Introduction

We consider the multiple scattering of time-harmonic ocean waves by a finite array of ice floes floating at the surface of a fluid domain with constant depth and infinite horizontal extent. Our goal is to model wave attenuation and directional spreading due to scattering in inhomogeneous ice-covered seas. In particular, we seek to understand better the effect of randomly perturbed arrays on these properties, which can be extracted from the scattering matrix of the array. We use the method proposed in [2] and extend the analysis of [1] for acoustic waves to discuss the effect of introducing random perturbations in a regular array of ice floes on the scattering properties of the array.

2 Preliminaries

Cartesian coordinates $\mathbf{x} = (x, y, z)$ are used in the fluid domain Ω with $z = 0$ and $z = -h$ coinciding with the undisturbed free surface and seabed, respectively. Let M denote the number of floes. All floes are circular (with radius a), have uniform thickness D and experience flexural motion when perturbed from rest. For each floe j, $1 \le j \le M$, we define the coordinates (x_j, y_j) of its centre and the local polar coordinates (r_j, θ_j). We assume that the centres of all floes are contained in the slab $0 \le x \le L$.

Within the framework of time-harmonic linear water wave and thin elastic plate theories, the potential $\Phi(\mathbf{x}, t) = \mathrm{Re}\{(g/\mathrm{i}\omega)\phi(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\}$ is used to describe the fluid motion, where ω is the radian frequency and g acceleration due to gravity. The (reduced) potential ϕ then satisfies

$$\left(\nabla^2 + \partial_z^2\right)\phi = 0 \ (\mathbf{x} \in \Omega) \quad (1)$$

$$\partial_z \phi = 0 \ (z = -h) \quad (2)$$

$$\partial_z \phi = \alpha\phi \ (r_j > a,\ z = 0) \quad (3)$$

$$\left(\beta\nabla^4 + 1 - \alpha d\right)\partial_z \phi = \alpha\phi \ (r_j < a,\ z = -d) \quad (4)$$

$$\partial_{r_j}\phi = 0 \ (r_j = a,\ -d < z < 0). \quad (5)$$

We have introduced the parameters $\alpha = \omega^2/g$, $\beta = ED^3/12(1-\nu^2)\rho g$, where $E \approx 6\,\mathrm{GPa}$ and $\nu \approx 0.3$ are ice Young's modulus and Poisson's ratio, respectively, $\rho \approx 1025\,\mathrm{kg\,m^{-3}}$ is the fluid density and $d = (\rho_i/\rho)D$ is the floe draught, with $\rho_i \approx 922.5\,\mathrm{kg\,m^{-3}}$ the ice density. Free edge conditions are further imposed at the floe edges, and the scattered field obeys a radiation condition in the far-field to ensure its decay.

3 Transmission matrix

We consider a wave forcing of the form

$$\phi^{\mathrm{In}}(\mathbf{x}) = \psi(z)\int_{-\pi/2}^{\pi/2} A(\tau)\mathrm{e}^{\mathrm{i}k(x\cos\tau + y\sin\tau)}\,\mathrm{d}\tau, \quad (6)$$

where $\psi(z) = \cosh k(z+h)/\cosh kh$ describes the vertical motion and k is the wavenumber related to frequency ω through the free surface dispersion relation $gk\tanh kh = \omega^2$. The incident field is a superposition of plane waves travelling at angle $\tau \in (-\pi/2, \pi/2)$ with amplitudes $A(\tau)$. The wave field transmitted through the slab is given in the form

$$\phi^{\mathrm{T}}(\mathbf{x}) = \psi(z)\int_{-\pi/2}^{\pi/2} B(\chi)\mathrm{e}^{\mathrm{i}k((x-L)\cos\chi + y\sin\chi)}\,\mathrm{d}\chi, \quad (7)$$

for $x > L$, such that the unknown transmitted spectrum $B(\chi)$ is related to the incident spectrum $A(\tau)$ through

$$B(\chi) = \int_{-\pi/2}^{\pi/2} \mathcal{T}(\chi : \tau)A(\tau)\,\mathrm{d}\tau. \quad (8)$$

The solution to the transmission problem is fully described by the transmitted kernel $\mathcal{T}(\chi : \tau)$.

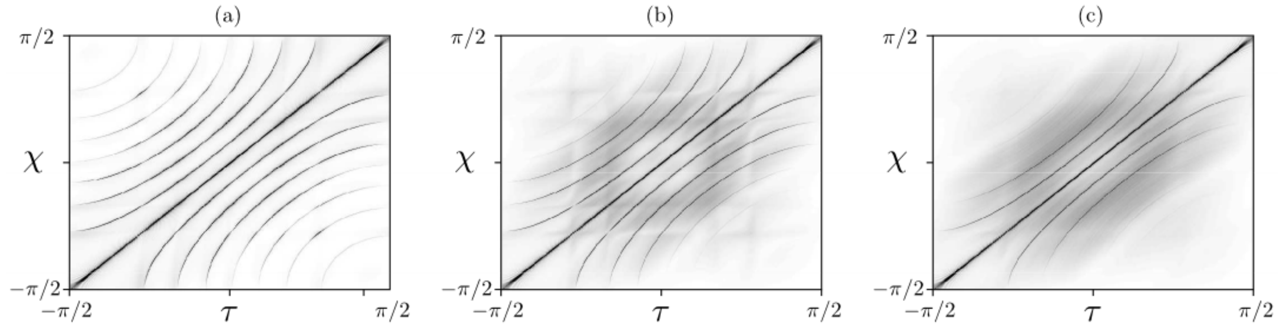

Figure 1: Magnitude of the transmission matrix for (a) $\mu = 0$, (b) 0.5 and (c) 1.

Similar relations exist for the reflected field, but are not discussed here.

A solution method was developed in [2] which provides semi-analytical expressions for the transmitted (and reflected) kernels. The method is summarised as follows:

1. derive a self-consistent multiple scattering solution using cylindrical multipole expansions to represent the field near each floe (based on Graf's addition theorem);
2. transform the multipole expansions into integrals of plane waves using Sommerfeld's integral representation of Hankel functions.

A numerical solution is then obtained by discretising $A(\tau)$, $B(\chi)$ and $\mathcal{T}(\chi : \tau)$ in (8) at N regular angular samples τ_i and χ_i, $1 \leq i \leq N$. Using the trapezoidal rule for numerical integration (8) becomes

$$\mathbf{b} = \mathbf{S}^T \mathbf{a}, \qquad (9)$$

where $\mathbf{S}^T$ is the transmission matrix, and $\mathbf{a}$ and $\mathbf{b}$ are discrete versions of A and B, respectively.

4 Results

We analyse the magnitude $\mathbf{S}^T$. The entry corresponding to $\mathcal{T}(\chi_i : \tau_j)$ describes the transmitted wave amplitude at angle $\chi = \chi_i$ due to an impulse incident at angle $\tau = \tau_j$, i.e. $A(\tau) = \delta(\tau - \tau_j)$ with δ denoting the Delta function. We consider an array of 51 floes regularly positioned along the y-axis with spacing $s = 300\,\text{m}$, such that $y_j = \tilde{y}_j = s(j - 26)$, for $1 \leq j \leq 51$. We set $h = 200\,\text{m}$, $a = 100\,\text{m}$, $D = 1.5\,\text{m}$ and the wave period $T = 7\,\text{s}$. We then introduce a random perturbation on the position of each floe, such that $(x_j, y_j) = (50\varepsilon\mu, \tilde{y}_j + 50\varepsilon\mu)$, where $0 \leq \mu \leq 1$ and ε is a random variable with uniform distribution in $[-1, 1]$.

Figure 1 shows a grayscale image of the magnitude of the entries of $\mathbf{S}^T$ for $\mu = 0$, 0.5 and 1. For the two latter cases, the entries are averaged over 100 random realisations of the array. The grayscale image shows clearly the existence of the scattering angles in the regular case ($\mu = 0$), which arise from scattering by regular gratings and satisfy $\sin\chi = \sin\tau + 2n\pi/ks$ for n integer. When perturbations are introduced ($\mu > 0$) and increase, the scattering angles are gradually filtered out and the energy is redistributed across the directional range (shaded areas). The dominant diagonal corresponds to a transmitted wave travelling in the same direction as the incident wave. With sufficient disorder, waves are expected to transmit with directionality unaffected but reduced amplitudes. These results are consistent with the findings of [1] for acoustic waves.

References

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